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To Thomas Stewart Wesner

Dad

To Margot

Phil


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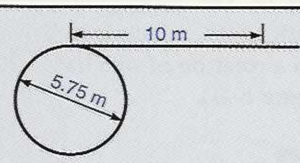
C H A P T E R

6 Radian Measure, Properties of the Trigonometric and Inverse Trigonometric Functions

We begin this chapter with a second system of angle measurement, called radians. We then study some additional properties of the trigonometric functions. At the end of the chapter another set of functions are presented—the inverse trigonometric functions.

6–1 Radian measure



The diameter of a wheel that moves the cable of a ski lift is 5.75 meters, as shown in the diagram. Through what angle, in degrees, does the wheel have to move to advance one of the chairs a distance of 10 meters?

Radian measure is a way to measure angles that is actually used more than degree measure in advanced science and engineering applications. It is useful for solving this problem, as well as many, many other situations.

The unit circle

The circle with radius one and center at the origin is described by the equation

$$x^2 + y^2 = 1$$

It is called the unit circle (see figure 6–1). Observe that the absolute values of the x- and y-coordinates of any point not on an axis describe the lengths of two sides of a right triangle with hypotenuse of length one. The Pythagorean theorem shows that for these points $x^2 + y^2 = 1$. Those points of the circle that are on an axis also satisfy this equation.

The circumference C of a circle with radius r is the distance around the circle. This distance is found using the relation $C = 2\pi r$. Since the radius r for the unit circle is one, its circumference is $C = 2\pi$ (about 6.28 units).

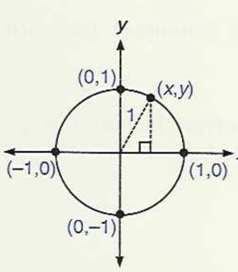


Figure 6–1

Note The constant π is approximately 3.14159. It is a much-used number, about which entire books have been written. It is an irrational number, and has been approximated to over a billion digits![1]

Radian measure

A second system of angle measurement uses units called **radians.** This system is used extensively in engineering and scientific applications, as well as in the calculus. We will use it throughout our study of trigonometry. To define this system of angle measurement we use the unit circle.

Let θ be an angle in standard position, and let s represent the distance from the point (1,0) along the circumference of the unit circle to the terminal side of θ. The distance s is called the **arc length** (see figure 6–2). As with degree measure if the distance is measured in a counterclockwise direction we say s is positive, and if in a clockwise direction s is negative.

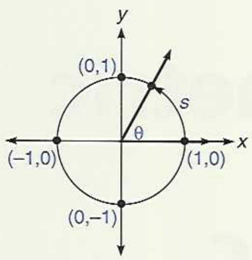


Figure 6–2

We define the radian measure of an angle to be this arc length s.

Radian measure of an angle in standard position

Let θ be an angle in standard position. Let s be the corresponding arc length on the unit circle. Let s be positive if measured in the counterclockwise direction, and negative if measured in the clockwise direction.

Then s is the radian measure of the angle θ.

For example, an angle of degree measure 180° has an arc length that corresponds to half the circumference of the unit circle. The corresponding radian measure is half of the circumference, or one half of 2π, which is π. Thus, the radian measure of an angle that corresponds to a rotation of one half a circle, in the counterclockwise direction, is π (see figure 6–3).

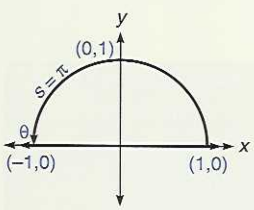


Figure 6–3

Conversions between radian and degree measure

Since 360° corresponds to a full revolution, and the circumference of the unit circle (2π) also corresponds to a complete revolution about the unit circle, the following relation is true:

$$\frac{\text{arc length } s}{\text{circumference } (2\pi)} = \frac{\text{measure of angle in degrees}}{360^\circ}$$

If we multiply each member by 2 we obtain the same true statement, but with smaller denominators of π and 180°.

[1]An interesting book on π is *A History of* π by Petr Beckmann, Golem Press, Boulder, Colo., 1977. Gregory V. and David V. Chudnovsky of Columbia University calculated 1,011,196,691 digits of π in 1989.

We use this proportion[2] to convert between degree and radian measure.

Radian/degree proportion

Let θ be an angle in standard position with degree measure $\theta°$ and radian measure s. Then,

$$\frac{s}{\pi} = \frac{\theta°}{180°}$$

The radian measure of an angle is a real number, defined with no units in mind. We often add the word radians after such a measure, but this is not necessary where it is clear that the real number refers to the measure of an angle. Observe that in the radian/degree proportion the ratio of degrees to degrees is unitless also. For example, $\frac{90°}{180°}$ is the same as the unitless ratio $\frac{1}{2}$.

We can describe the measure of an angle in standard position by using degree measure or by stating the arc length to which the angle corresponds on the unit circle. The proportion just described shows the relationship between these two systems.

■ ***Example 6–1 A***

Compute the radian or degree measure, given the measure for each angle in degrees or radians.

1. $-210°$

$$\frac{s}{\pi} = \frac{-210°}{180°} \qquad \text{Replace } \theta° \text{ with } -210°$$

$$s = \frac{-210(\pi)}{180} = -\frac{7\pi}{6}$$

Therefore $-210°$ corresponds to $-\frac{7\pi}{6}$.

2. $\frac{7\pi}{5}$

$$\frac{\frac{7\pi}{5}}{\pi} = \frac{\theta°}{180°} \qquad \text{Replace } s \text{ by } \frac{7\pi}{5}$$

$$\frac{7\pi}{5} \cdot \frac{1}{\pi} = \frac{\theta°}{180°} \qquad \text{Division by } \pi \text{ is the same as multiplication by } \frac{1}{\pi}$$

$$\frac{7}{5} \cdot 180° = \theta° \qquad \text{Multiply each member by } 180°$$

$$252° = \theta°$$

$\frac{7\pi}{5}$ (radians) corresponds to $252°$.

[2]A proportion is a statement of equality between two ratios (fractions).

3. 1

Note that this means 1 radian.

$$\frac{1}{\pi} = \frac{\theta°}{180°}$$

$$\frac{180°}{\pi} = \theta°$$ Multiply each member by 180°

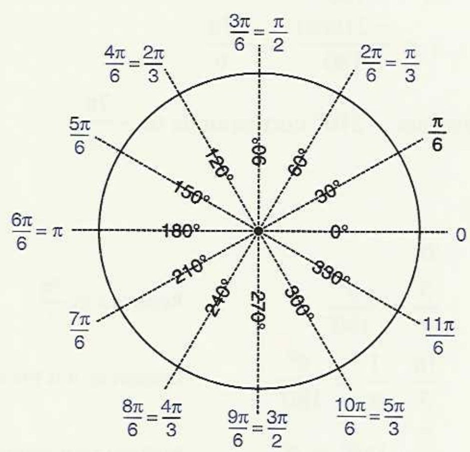
$$57.30° \approx \theta°$$ Decimal approximation to $\frac{180}{\pi}$

Thus, one radian corresponds to $\frac{180°}{\pi}$ or about 57.30°. ■

Note It is useful to remember that 1 radian is a little less than 60°, and that 2π radians exactly equals 360°.

Common radian measures

The unit circle can be very helpful in getting a feeling for radian measure. Those values of radian measure that correspond to quadrantal angles (0°, 90°, 180°, etc.) and to angles with reference angles of 30°, 45°, and 60° are common. In particular, the following correspondences are useful: $\frac{\pi}{6}$ and 30°, $\frac{\pi}{4}$ and 45°, and $\frac{\pi}{3}$ and 60°. The unit circle can be conveniently marked in terms of multiples of $\frac{\pi}{6}$ radians (30°) and of multiples of $\frac{\pi}{4}$ radians (45°). This is shown in figure 6–4.



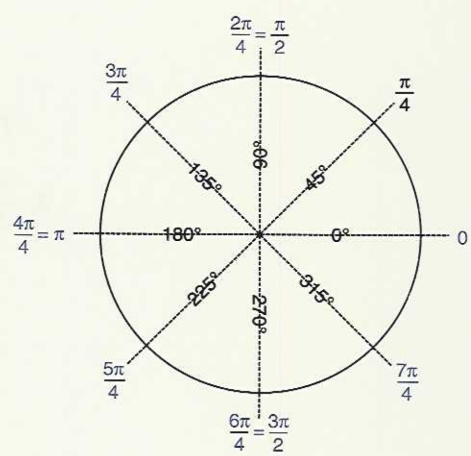


Figure 6–4

Using the unit circle to find values of trigonometric functions

Recall that if (x,y) is a point on the terminal side of an angle θ (in standard position), and $r = \sqrt{x^2 + y^2}$, then $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$. On the unit circle, $r = 1$, and therefore if (x,y) is the point on the unit circle that intersects the terminal side of θ, then $\sin\theta = y$, $\cos\theta = x$. Figure 6–5 shows this fact.

Figure 6–5 is very useful to keep handy because it shows the degree and radian measure for many common angles with measure between 0° and 360° (0 and 2π in radian measure). The angles shown are either quadrantal or have reference angles of measure 30° $\left(\frac{\pi}{6}\right)$, 45° $\left(\frac{\pi}{4}\right)$, or 60° $\left(\frac{\pi}{3}\right)$. The figure also shows the point on the terminal side of an angle where it meets the unit circle. As stated, the (x,y) pair is $(\cos\theta, \sin\theta)$. Note that the radian measure is shown for the values 1, 2, 3, . . . , 6 as well. For example, 2 (radians) is near $\frac{2\pi}{3}$ (radians), or 120°, because $\frac{2\pi}{3} \approx 2.1$.

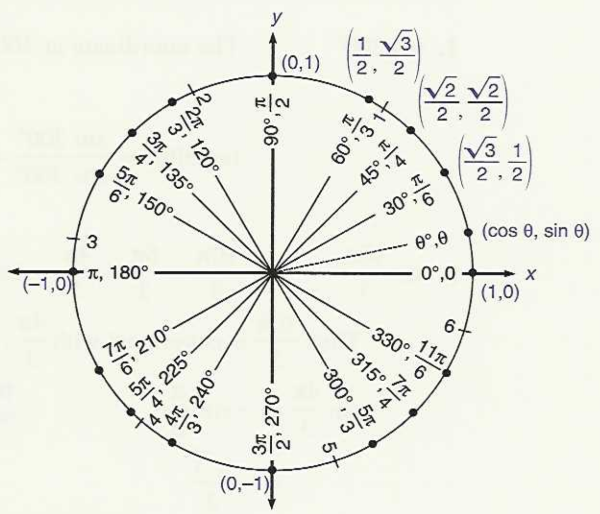


Figure 6–5

Using symmetries in the unit circle

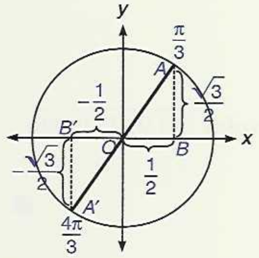


Figure 6–6

Observe that you can find the sine or cosine value for any of the angles shown by observing the symmetries in figure 6–5. For example, the coordinates at $\frac{4\pi}{3}$ must be $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The reason is shown in figure 6–6. Triangle ABO is congruent (same size and shape) to triangle $A'B'O$. This can be shown geometrically because angle AOB is the same size as angle $A'OB'$, and the hypotenuse of each triangle is the same length.

Similarly, the coordinate at $\frac{5\pi}{6}$ (figure 6–5) must have the same y-coordinate as at $\frac{\pi}{6}$, but the opposite of the x-coordinate. Thus the coordinates there must be $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, and from this point we know that $\sin \frac{5\pi}{6} = \sin 150° = y = \frac{1}{2}$.

Calculators and radian mode

Calculators are programmed to accept angle input in radian measure. All scientific calculators have a key, often marked $\boxed{\text{DRG}}$ or $\boxed{\text{MODE}}$, to tell the calculator to accept angles in radian measure. On the TI-81 use the $\boxed{\text{MODE}}$ key and darken the Rad mode, then use $\boxed{\text{ENTER}}$ to make the change. For angles that are not coterminal with those in figure 6–5 use the calculator.

■ ***Example 6–1 B***

Find the required value. Use figure 6–5 as an aid when possible; otherwise use a calculator and round the answer to four decimal places.

1. tan 300° The coordinate at 300° is $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

$$\tan 300° = \frac{\sin 300°}{\cos 300°} = \frac{y}{x} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}.$$

2. $\sin \frac{10\pi}{3}$ $\frac{10\pi}{3} = \frac{6\pi}{3} + \frac{4\pi}{3} = 2\pi + \frac{4\pi}{3}$

Thus $\frac{10\pi}{3}$ is coterminal with $\frac{4\pi}{3}$, so $\sin \frac{10\pi}{3} = \sin \frac{4\pi}{3}$.

$$\sin \frac{4\pi}{3} = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

The points associated with these angles are diagonally opposite on the circle

Thus, $\sin \frac{10\pi}{3} = -\frac{\sqrt{3}}{2}$.

3. sin 1.2

Make sure the calculator is in radian mode.

1.2 $\boxed{\text{sin}}$ Display $\boxed{0.932039086}$

TI-81: $\boxed{\text{SIN}}$ 1.2 $\boxed{\text{ENTER}}$

$\sin 1.2 \approx 0.9320$ ■

Coterminal and reference angles in radian measure

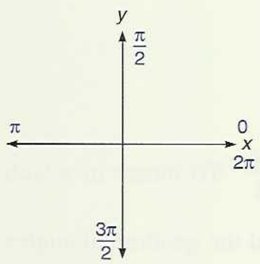


Figure 6–7

Figure 6–7 shows the smallest positive radian measure of the quadrantal angles, as well as the fact that a full revolution (circle) can be described by 2π radians. Observe that the quadrantal angles 0°, 90°, 180°, 270°, and 360° are, in radians, 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π.

In degree measure, all angles that differ in measure by integer multiples of 360° are coterminal. For radian measure the difference is multiples of 2π. If k is an integer (positive, zero, or negative), then integer multiples of 2π are $k \cdot 2\pi$, or $2k\pi$.

> **Coterminal angles, radian measure**
> Two angles α and β are coterminal if $\alpha = \beta + 2k\pi$, k an integer.

Reference angles are found in the same manner as with degree measure (section 5–4) except that 180° becomes π and 360° becomes 2π. If the measure of θ in radians is positive and less than 2π, the following rules give the value of θ', the reference angle.

Quadrant in which θ terminates	Value of θ', the reference angle
I	$\theta' = \theta$
II	$\theta' = \pi - \theta$
III	$\theta' = \theta - \pi$
IV	$\theta = 2\pi - \theta$

The reference angle/ASTC procedure for radian measure

The reference angle/ASTC procedure (section 5–4) can also be used instead of figure 6–5. It is restated here. Table 6–1 is the same as Table 5–1 except that the radian measure of each angle is included. Figure 6–8 is the same as figure 5–13 except that the quadrantal angles less than 2π (360°) are shown in radian measure.

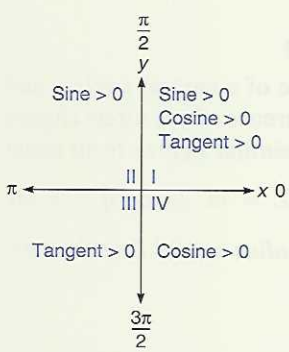


Figure 6–8

	Sine	Cosine	Tangent
$\frac{\pi}{6}$, 30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$, 45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$, 60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Table 6–1

> **Reference angle/ASTC procedure**
> To find the value of a trigonometric function for a nonquadrantal angle whose reference angle is $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$:
>
> 1. Find the value of the reference angle.
> 2. Find the value of the appropriate trigonometric function for the reference angle from Table 6–1.
> 3. Determine the sign of this value using the ASTC rule (figure 6–8).

In applying this method it is very important to know in which quadrant the angle terminates. We need this information to be able to find the reference angle and to apply the ASTC rule. Example 6–1 C illustrates this process.

■ ***Example 6–1 C***

Use table 6–1 and the ASTC rule to find $\sin \frac{11\pi}{3}$

$$\frac{11\pi}{3} = \frac{6\pi}{3} + \frac{5\pi}{3} = 2\pi + \frac{5\pi}{3}$$

Thus $\frac{11\pi}{3}$ is coterminal with $\frac{5\pi}{3}$, so $\sin \frac{11\pi}{3} = \sin \frac{5\pi}{3}$. To locate in which quadrant the angle $\frac{5\pi}{3}$ terminates, rewrite $\frac{5\pi}{3}$ as $\frac{10\pi}{6}$ and the quadrantal angles in terms of denominators of 6.

$$0, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2}, \quad 2\pi$$

$$\frac{0}{6}, \quad \frac{3\pi}{6}, \quad \frac{6\pi}{6}, \quad \frac{9\pi}{6}, \quad \frac{12\pi}{6}$$

Now we see that $\frac{9\pi}{6} < \frac{10\pi}{6} < \frac{12\pi}{6}$ so $\frac{10\pi}{6} = \frac{5\pi}{3}$ is in quadrant IV.

Step 1: $\theta' = 2\pi - \frac{5\pi}{3} = \frac{12\pi}{6} - \frac{5\pi}{3} = \frac{\pi}{3}$ In quadrant IV, θ' is $360° - \theta$ or $2\pi - \theta$

Step 2: $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ Table 6–1

Step 3: $\sin \frac{5\pi}{3} < 0$ ASTC rule; $\sin \theta < 0$ in quadrant IV

Thus, $\sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$ and therefore $\sin \frac{11\pi}{3} = -\frac{\sqrt{3}}{2}$. ■

Arc length and radian measure

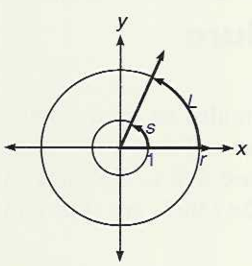
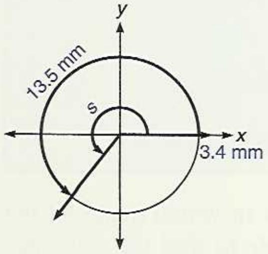


Figure 6–9

A simple relation exists between the radian measure of a central angle s and arc length L determined by that angle on the circumference of any circle (figure 6–9). Geometry tells us that corresponding parts of similar figures form equal ratios. This means, in this case, that $\frac{s}{1} = \frac{L}{r}$, or $L = rs$. Thus, *if s is the radian measure of a central angle on a circle of radius r, and L is the corresponding arc length,* then

$$L = rs$$

■ ***Example 6–1 D***



Use the relation $L = rs$ to solve each problem.

1. Find the measure in radians of the central angle corresponding to an arc length of 13.5 mm on a circle of diameter 6.8 mm.

$r = 3.4$ mm One half the diameter

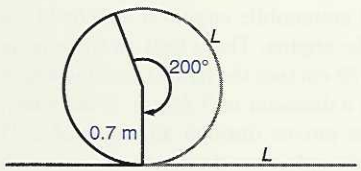
$L = rs$

$13.5 \text{ mm} = (3.4 \text{ mm})s$ Substitute known values

$\frac{13.5}{3.4} = s$

$3.97 \approx s$ Rounding to nearest 0.1

Thus, the central angle measures 3.97 radians.



2. A railroad car has wheels with diameter 1.4 m (meters). If the wheels move through an angle of 200°, how far does the train move?

As illustrated, the distance the train will move is the same as the arc length L on the wheel. This length is determined by the central angle of 200°. We will find the measure of the central angle θ in radians, then use the relation $L = rs$.

$$\frac{200°}{180°} = \frac{\theta}{\pi}\text{, so } \frac{10\pi}{9} = \theta$$

$$L = rs$$

$$L = 0.7\left(\frac{10\pi}{9}\right) \qquad \text{The radius } r \text{ is half the diameter of 1.4 m; } s = \theta \text{ (in radians)}$$

$$L \approx 2.4 \text{ m}$$

Thus, the train moves 2.4 meters when the wheels move through an angle of 200°. ■

Mastery points

Can you

- Convert between degree and radian measure for angles?
- Find the exact value of a trigonometric function for an angle whose reference angle is $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$ using figure 6–5 or the reference angle/ASTC procedure?
- Use the relation $L = rs$ to solve problems concerning arc length on any circle?

Exercise 6–1

1. State the algebraic relation (equation) that describes the unit circle.

Convert the following degree measures to radian measure. Leave your answers in both exact form and approximated to two decimal places.

2. 30° **3.** 45° **4.** 60°
5. 100° **6.** −200° **7.** −300°
8. −135° **9.** 270° **10.** 750°
11. 127° **12.** −422° **13.** −305°

Convert the following radian measures into degree measure. Leave answers in both exact form and approximated to two decimal places.

14. $\frac{5\pi}{2}$ **15.** $\frac{11\pi}{6}$ **16.** $\frac{2\pi}{7}$
17. $\frac{3\pi}{5}$ **18.** $\frac{10\pi}{9}$ **19.** $\frac{2\pi}{9}$
20. $-\frac{5\pi}{3}$ **21.** $-\frac{17\pi}{6}$ **22.** $-\frac{5\pi}{7}$
23. $\frac{3}{2}$ **24.** $\frac{11}{6}$ **25.** $-\frac{12}{17}$
26. 1.5 **27.** 2 **28.** 3.25
29. −5 **30.** −6

Find the following function values where the angle is given in radian measure. Round your answers to four decimal places.

31. sin 0.9	**32.** cos 1.1	**33.** tan 0.5
34. sec 1.4	**35.** csc 0.7	**36.** cot 1.5
37. sin 2.3	**38.** cos 3.5	**39.** tan 4.1
40. sec 5.2	**41.** csc 2.5	**42.** cot 1.9

Find the exact function values for the following angles.

43. $\sin \frac{2\pi}{3}$	**44.** $\tan \frac{5\pi}{4}$	**45.** $\cos \frac{11\pi}{6}$
46. $\tan \frac{5\pi}{6}$	**47.** $\cos \frac{4\pi}{3}$	**48.** $\sin \frac{5\pi}{6}$
49. $\sin \frac{5\pi}{4}$	**50.** $\cos \frac{7\pi}{4}$	**51.** $\tan \frac{5\pi}{3}$

Solve the following problems.

52. Find the length of the arc determined by a central angle of 2.1 (radians) on a circle of diameter 10 inches.

53. Find the measure, in radians, of a central angle on a circle of radius 4.5 mm (millimeter) determined by an arc length of 12 mm, to the nearest 0.1 radian.

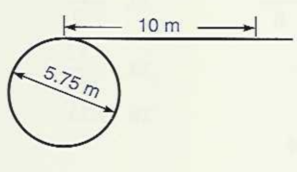
54. Find the length of the arc determined by a central angle of 300° on a circle of diameter 12 mm.

55. Find the length of the arc determined by a central angle of 45° on a circle of radius 8.3 inches, to the nearest 0.1 inch.

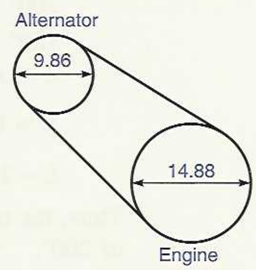
56. Find the measure, in both radians and degrees, of the central angle determined by an arc length of 14.5 mm on a circle with diameter 10.3 mm. Round both answers to the nearest tenth.

57. The diameter of a wheel on an automobile is 32.4 inches. If the wheel moves through an angle of 85°, how far will the car move?

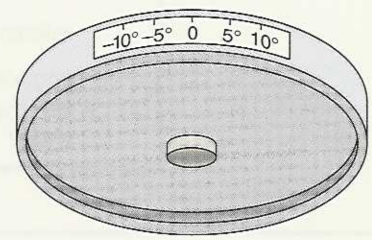
58. The diameter of a wheel that moves the cable of a ski lift is 5.75 meters, as shown in the diagram. Through what angle, in degrees, does the wheel have to move to advance one of the chairs a distance of 10 meters?



59. The alternator on an automobile engine is attached by a belt to a wheel on the engine. The wheel on the engine has a diameter of 14.88 cm (see the figure), and the wheel on the alternator has a diameter of 9.86 cm. If the wheel driven by the engine moves through an angle of 2.85 radians, through what angle does the alternator move, to the nearest 0.01 radian?



60. A decal is being made to indicate timing marks on a wheel attached to the front of an engine (see the diagram). The radius of the wheel is 86.6 mm. What should the distance be between the $-10°$ and $10°$ marks, to the nearest millimeter?



61. In a certain series circuit the applied voltage V, in volts, is determined by the function

$$V = 200 \sin(35t + 1)$$

where t represents time in milliseconds and the expression $35t + 1$ is in radians. Compute V to the nearest 0.1 volt for the following values of t:
a. 0 **b.** 0.1 **c.** 0.8 **d.** 1

62. An equation that arises in finding the trajectory of a rocket is $r = \dfrac{p}{1 + e\cos(s - C)}$. Assume $p = 200$, $e = 1.5$, and $C = 0.5$. Find r if **a.** $s = 1$ **b.** $s = 1.25$

63. The position d at the end of a spring, under certain initial conditions, as a function of time t in seconds, is $d = \frac{1}{3}\cos 8t - \frac{1}{4}\sin 8t$. Compute d to the nearest thousandth for the following values of t: **a.** $\frac{1}{8}$ **b.** $\frac{1}{4}$

64. An equation that can be used to compute sin x, if x is in radians, is called the *Maclaurin series* for the sine function. It is $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$, where

$$3! = 1 \cdot 2 \cdot 3 = 6$$
$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$
$$7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5{,}040, \text{ etc.}$$

($n!$ is read "n factorial" and is defined as $1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n$.) Although the Maclaurin series goes on forever, good accuracy is obtained by using the first few terms. Use the first four of the five terms shown to compute approximations to

a. sin 0.1 **b.** sin 0.5 **c.** sin 1 **d.** $\sin \frac{\pi}{6}$

Check the results with the sine key of a calculator. (Some computers use a method similar to this for computing the trigonometric functions.)

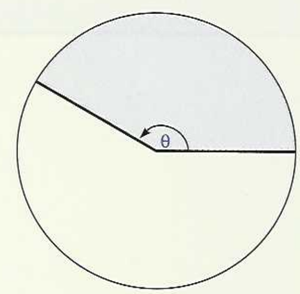
65. (See problem 64.) The Maclaurin series for the cosine function, if x is in radians, is $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$. Use the first four of the five terms shown in problem 64 to calculate approximations to

a. cos 0.8 **b.** cos 1 **c.** cos 1.3

d. Approximate cos 10° by first converting 10° to radians.

The following discussion applies to problems 66–75. The area A of a circle is determined by the equation $A = \pi r^2$. The area of a sector of a circle (the shaded portion shown in the diagram) is proportional to the measure of the central angle determining the sector. Thus, the area of a sector determined by an angle θ with measure s (radians) or θ° is

$$\frac{s}{2\pi}A \text{ or } \frac{\theta°}{360°}A$$



Substituting πr^2 for A in each case, and letting A_p mean the area of a sector (p stands for "part"), produces

$$A_p = \frac{s}{2}r^2, \text{ or } A_p = \frac{\theta°}{360°}\pi r^2$$

for the angle measured in radians and degrees, respectively. Note that these expressions only make sense if $0 \le s \le 2\pi$ or $0° \le \theta° \le 360°$. Find the area of the sector determined by each of the following angles and radii. Give both the exact answer and a two-decimal-place approximation.

66. θ° = 35°, r = 6 inches

67. θ° = 140°, r = 7 mm

68. s = 4, r = 10 mm

69. s = 2.4, r = 5 inches

70. $s = \frac{5}{12}$, r = 6 inches

71. $s = \frac{2\pi}{5}$, r = 8 inches

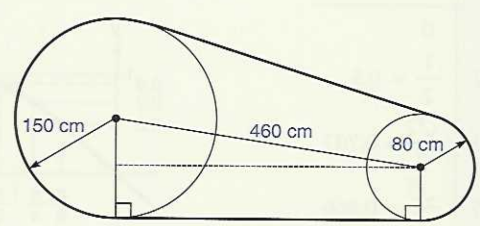
72. θ° = 100°, r = 10 inches

$\boxed{73.}$ θ° = 15°, r = 9 mm

74. s = 3.25, r = 44.6 mm

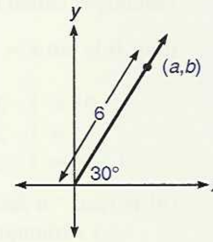
75. Find the central angle θ necessary to form a sector of area 14.6 cm² on a circle of radius 4.85 cm. Find the angle both in radians and degrees, to two decimal places.

76. The figure shows two wheels, with a thin belt wrapped around both wheels. The larger wheel has a radius of 150 cm, and the smaller has a radius of 80 cm. The wheels are 460 cm apart. Ignoring the thickness of the belt, calculate the total length of the belt.



Skill and review

1. Simplify the expression $\dfrac{\csc x - 1}{\csc x}$ (no fraction should appear in the result).
2. Solve the literal equation $A = (n_1 - 1) + b(n_2 - n_1)$ for n_2.
3. Solve the literal equation $A = (n_1 - 1) + b(n_2 - n_1)$ for n_1.
4. Find the coordinates of the point (a,b) in the figure. Leave the values in exact form (do not put the results in decimal form).
5. Solve the inequality $\left|\frac{1}{2}x + 10\right| > 8$.



6–2 Properties of the sine, cosine, and tangent functions

The activity of sunspots seems to follow an 11-year cycle. Assuming that this activity can be roughly modeled with a sine wave, construct a sine function with period $\frac{360}{11}$°, amplitude 1, phase shift 90°, and vertical translation 2.

To understand many applications of the trigonometric functions, such as the one in this problem, it is necessary to have a good understanding of the properties of these functions. As with any functions we can gain a great deal of knowledge from their graphs.

In more advanced work we use radian measure more than degree measure. Radian measure is stressed for the remainder of this chapter.

Graph and properties of the sine function

Figure 6–10 is the graph of $y = \sin x$ for x (in radians) between 0 and 2π. This graph can be obtained by plotting points for various values of x. The points for the table of values shown are plotted in the figure. When we allow x to take on any value, this graph is repeated over and over. This is because for $x > 2\pi$ or $x < 0$ we have angles that are coterminal with values we have already plotted. Every 2π units (once around the unit circle) we find that the graph repeats itself.

To obtain the graph with the TI-81 calculator, first make sure the calculator is in radian mode. See section 6–1.

x	$\sin x$
0	0
$\frac{\pi}{6} \approx 0.52$	$\frac{1}{2} = 0.5$
$\frac{\pi}{4} \approx 0.79$	$\frac{\sqrt{2}}{2} \approx 0.707$
$\frac{\pi}{3} \approx 1.05$	$\frac{\sqrt{3}}{2} \approx 0.866$
$\frac{\pi}{2} \approx 1.57$	1

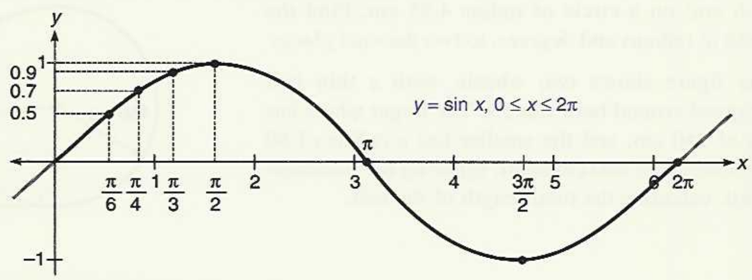


Figure 6–10

To graph the function, do the following:

$\boxed{\text{ZOOM}}$ 7 Trig (basic scale settings for many trigonometric functions)
$\boxed{\text{RANGE}}$ Change Xmin to 0, Ymin = −2, Ymax = 2, Yscl − .5
$\boxed{\text{Y=}}$ $\boxed{\text{SIN}}$ $\boxed{\text{X|T}}$ $\boxed{\text{GRAPH}}$

We can use the graph in figure 6–10 to produce the graph in figure 6–11, which shows the graph of the sine function for all values of x. The graph in figure 6–10 is one *cycle* of the sine function. We refer to figure 6–10 as the **basic sine cycle.**

The repetitious nature of the sine function can be described with the identity

$$\sin x = \sin(x + k \cdot 2\pi), \quad k \text{ any integer}$$

We say that the *sine function is 2π-periodic,* because it repeats every 2π units. We also say that the **amplitude** of the sine function is 1. We define the amplitude of a periodic function like the sine function as half the difference of its greatest and least values.

The graph in figure 6–11 shows that the domain of the sine function is all real numbers, while the range (the y-values) is restricted to the interval from -1 to 1. This and other information is summarized later in table 6–2.

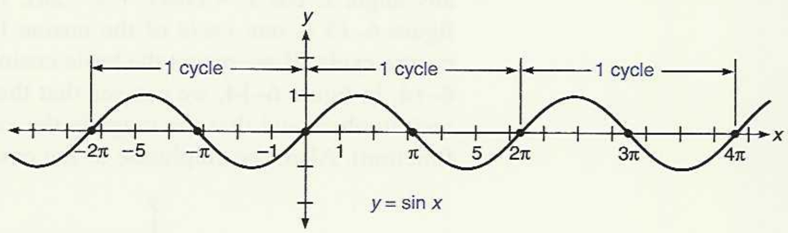


Figure 6–11

Another important point is that $\sin(-x) = -\sin x$ for any value of x. This is illustrated in figure 6–12, where we see that if we go equal distances in the positive and negative directions along the x-axis, the value of the sine function at each place is of the same magnitude (absolute value) but of the opposite sign. *Any function for which $f(-x) = -f(x)$ is true for all x in its domain is called an **odd function.*** Thus, *sine is an odd function.* Recall that the even/odd property of functions was introduced in section 3–5. We noted there that the graph of an odd function has symmetry about the origin. Example 6–2 A illustrates one application of the odd function property.

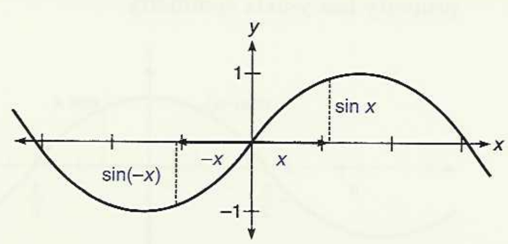


Figure 6–12

Graph and properties of the cosine function

The table shows some values of x and $\cos x$. Plotting these and other ordered pairs (x,y) where $y = \cos x$ (x in radians) and then connecting them with a smooth curve produces the graph shown in figure 6–13, for $0 \le x \le 2\pi$. Of course a graphing calculator could also be used to obtain this graph. The steps would be practically the same as those previously shown for graphing $y = \sin x$.

x	$\cos x$
0	1
$\frac{\pi}{6} \approx 0.52$	$\frac{\sqrt{3}}{2} \approx 0.866$
$\frac{\pi}{4} \approx 0.79$	$\frac{\sqrt{2}}{2} \approx 0.707$
$\frac{\pi}{3} \approx 1.05$	$\frac{1}{2} = 0.5$
$\frac{\pi}{2} \approx 1.57$	0
π	-1

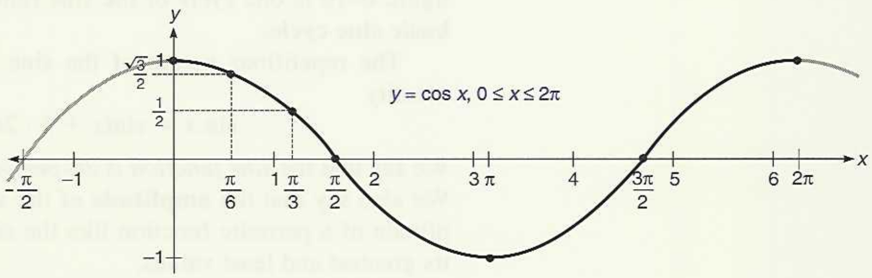


Figure 6–13

Just as with the sine function, *the cosine function is 2π-periodic.* Thus for any angle x, $\cos x = \cos(x + k \cdot 2\pi)$, where k is any integer. The graph in figure 6–13 is one *cycle* of the cosine function. We refer to it as the **basic cosine cycle.** If we repeat the basic cosine cycle we obtain the graph of figure 6–14. In figure 6–14, we can see that the domain of the cosine function is all real numbers and that the range is the values from -1 to 1 (as with the sine function). Also, the amplitude of the cosine function is 1.

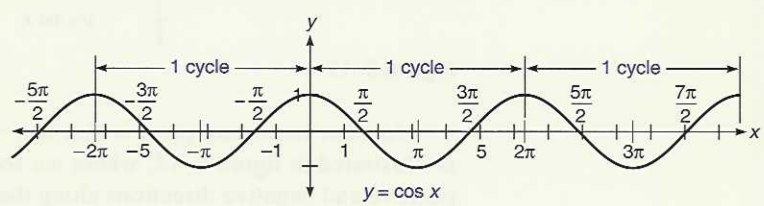


Figure 6–14

The graph in figure 6–14 also shows that $\cos(-x) = \cos x$ for all x. This is further illustrated in figure 6–15. *Any function for which $f(-x) = f(x)$ for all x is called an **even function,*** so the *cosine function is an even function.* Recall from section 3–5 that the graph of a function with the even function property has y-axis symmetry.

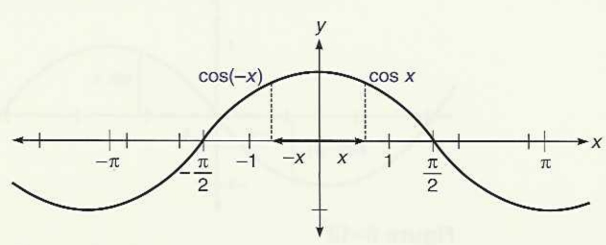


Figure 6–15

Finally, *the graphs of the sine and cosine functions have exactly the same shape.* Either one is identical to the other if it is shifted right or left by a suitable amount. The smallest amount is $\frac{\pi}{2}$, which is described in the statement that $\sin\left(x + \frac{\pi}{2}\right) = \cos x$. This statement can be proved with methods shown in section 7–2.

Graph and properties of the tangent function

To obtain the graph of the tangent function we can compute values and plot points. Some values of $\tan x$ for $0 \le x < \frac{\pi}{2}$ are shown in the table, and the graph of the tangent function is shown in figure 6–16.

x	$\tan x$
0	0
$\frac{\pi}{6} \approx 0.52$	$\frac{\sqrt{3}}{3} \approx 0.6$
$\frac{\pi}{4} \approx 0.79$	1
$\frac{\pi}{3} \approx 1.05$	$\sqrt{3} \approx 1.7$
1.25	≈ 3.0
1.50	≈ 14.1
1.55	≈ 48.1
$\frac{\pi}{2} \approx 1.57$	undefined

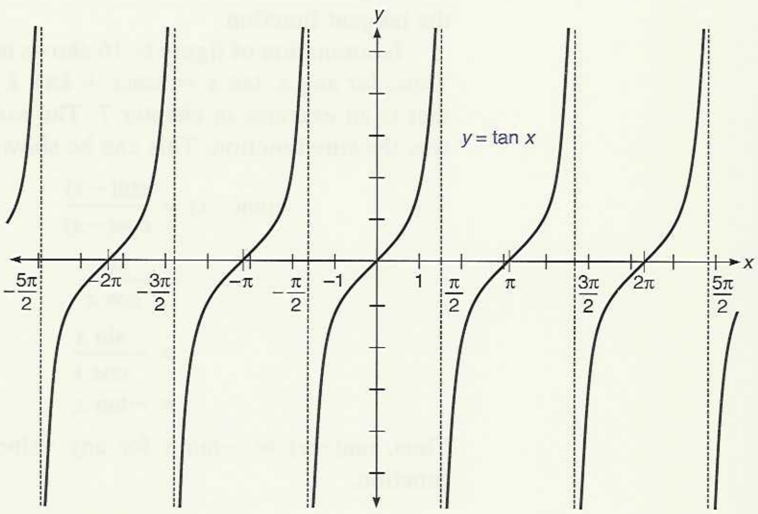


Figure 6–16

Observe that as x gets closer and closer to certain values, such as $\frac{\pi}{2}$, $|\tan x|$ gets larger and larger. At these values $\sin x \approx \pm 1$, so near these values $|\tan x| = \left|\frac{\sin x}{\cos x}\right| \approx \frac{1}{|\cos x|}$. Near these same points $|\cos x|$ is approaching 0. The reciprocal of a very small number is a very large number, so $\frac{1}{|\cos x|}$ gets larger and larger.

To obtain the graph above on the TI-81 use $\boxed{\text{RANGE } -8,8,-5,5}$, Xscl $= 1.5708 \approx \frac{\pi}{2}$, Yscl $= 1$. It will not look as nice, and the vertical, dashed lines (discussed below) will be solid. They are actually not part of the graph, and it is only the technical limitations of the calculator that causes them to be drawn at all.

If we recall that $\tan\theta = \frac{y}{x}$, where (x,y) is a point on the terminal side of angle θ, and if we examine figure 6–5, we will see that $x = 0$ at $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$, ... Thus, $\tan\theta$ is not defined for these angles, which can be described as $\frac{\pi}{2} + k\pi$, k an integer. Figure 6–16 shows vertical dashed lines at these points. These lines are **vertical asymptotes** of the tangent function. The tangent function is defined for all other values. The vertical asymptotes are lines that the tangent function approaches more and more closely as x approaches those values where the tangent function is undefined.

Because the tangent function gets arbitrarily large or small, the range of the tangent function is all real numbers. We do not define an amplitude for the tangent function.

Examination of figure 6–16 shows that *the tangent function is π-periodic.* Thus, for any x, $\tan x = \tan(x + k\pi)$, k any integer. The actual proof of this fact is an exercise in chapter 7. The *tangent function is an odd function,* as was the sine function. This can be shown algebraically as follows:

$$\begin{aligned}
\tan(-x) &= \frac{\sin(-x)}{\cos(-x)} && \text{Identity}\\
&= \frac{-\sin x}{\cos x} && \text{Sine is odd, cosine is even}\\
&= -\frac{\sin x}{\cos x}\\
&= -\tan x.
\end{aligned}$$

Thus, $\tan(-x) = -\tan x$ for any value x, and therefore tangent is an odd function.

Summary of properties

Table 6–2 summarizes the properties of the three functions we have examined. The unit circle (figure 6–5), the properties found in table 6–2, and the graphs of these three functions, have many applications. Some are illustrated in example 6–2 A.

Function	Domain	Range	Period
$y = \sin x$	R	$-1 \le y \le 1$	2π
$y = \cos x$	R	$-1 \le y \le 1$	2π
$y = \tan x$	$x \neq \frac{\pi}{2} + k\pi,\ k \in J$	R	π

$\sin(-x) = -\sin x$ (odd, origin symmetry)
$\cos(-x) = \cos x$ (even, y-axis symmetry)
$\tan(-x) = -\tan x$ (odd, origin symmetry)

Table 6–2

■ *Example 6–2 A*

Use table 6–2 and the graphs of the sine, cosine, and tangent functions to solve the following problems.

1. Find $\sin\left(-\frac{\pi}{3}\right)$

$$\sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} \qquad \text{Sine is an odd function, so } \sin(-x) = -\sin(x)$$
$$= -\frac{\sqrt{3}}{2}$$

2. Test the function $f(x) = x \sin x$ for the even/odd property. State which type of symmetry the graph of this function would have based on being even, odd, or neither even nor odd.

$$f(x) = x \sin x$$
$$f(-x) = (-x)\sin(-x)$$
$$= (-x)(-\sin x) \qquad \sin(-x) = -\sin x$$
$$= x \sin x$$

Thus, $f(x) = f(-x)$, so this function has the even function property. Its graph would therefore have symmetry about the y-axis.

3. Describe all points for which $\cos x = -\frac{1}{2}$.

Examination of the unit circle (figure 6–5) shows there are two points on $0 \le x < 2\pi$ for which $\cos x = -\frac{1}{2}$. These are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. This can also be seen in figure 6–14. Since the cosine function is 2π-periodic, $\cos x = -\frac{1}{2}$ for $x = \frac{2\pi}{3} + 2k\pi$ or $\frac{4\pi}{3} + 2k\pi$, $k \in J$. ■

Linear transformations of the sine and cosine functions

Most applications of the sine, cosine, and tangent functions require that they be transformed in some way to fit measured data or describe theoretical properties. In the remainder of this section, we examine linear transformations of these functions. Graphically these transformations represent moving the graph horizontally or vertically, or squeezing or expanding the graph horizontally or vertically.

Vertical scaling factors and translations

We first discuss *vertical scaling factors and translations.* We know from section 3–4 that, given the graph of $y = f(x)$, then the graph of $y = k \cdot f(x)$ is a vertically scaled version of the graph of $f(x)$.

Thus, the graph of $y = 2 \sin x$ is the same as the graph of $y = \sin x$, but vertically expanded 2 units. This is shown in figure 6–17. Similarly, the graph of $y = -4 \cos x$ is the graph of $y = \cos x$ but expanded vertically by a factor of 4 *and* reversed about the x-axis, because the coefficient -4 is negative.

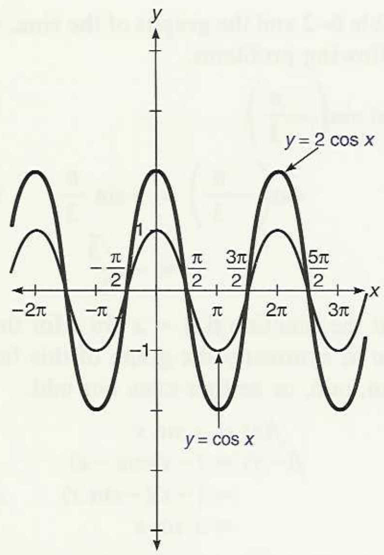


Figure 6–17

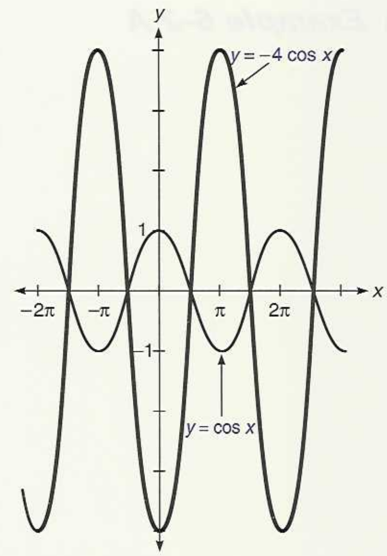


Figure 6–18

This is shown in figure 6–18. In general the graphs of $y = A \sin x$ and $y = A \cos x$, $A \in R$ are scaled vertically by the factor $|A|$ and the graph is reflected about the horizontal axis if $A < 0$.

We know (section 3–4) that the graph of $y = f(x) + k$ is the graph of $y = f(x)$ but shifted vertically by k units. Thus, for example, the graph of $y = \sin x + 3$ (not to be confused with $y = \sin(x + 3)$) is the graph of $y = \sin x$ but shifted vertically by 3 units. This is illustrated in figure 6–19. Thus, the graphs of $y = \sin x + D$ and $y = \cos x + D$ are the graphs of $y = \sin x$ or $y = \cos x$, shifted vertically D units.

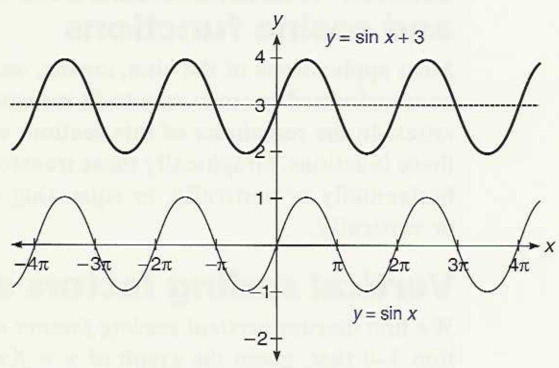
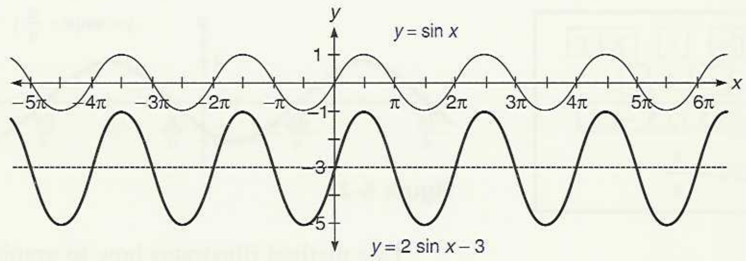


Figure 6–19

■ ***Example 6–2 B*** Graph the function $y = 2 \sin x - 3$.

This is the graph of $y = \sin x$ but shifted down 3 units, and with amplitude 2. This is shown in the figure.



Y=	2	SIN	X|T	−	3	RANGE −16,16,−6,2	Xscl = 1.5708

Horizontal scaling factors and translations

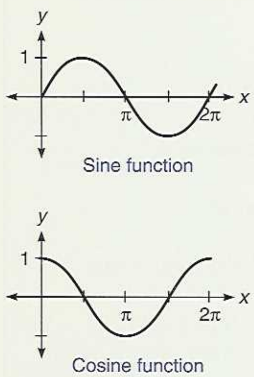


Figure 6–20

The **argument** of a function is the expression that represents the domain element. In $y = \sin x$, the argument is x. In $y = \cos 3x$ the argument is $3x$. In $y = \tan\left(x - \frac{\pi}{2}\right)$ the argument is $x - \frac{\pi}{2}$.

Now consider what we know about the sine and cosine functions. *As the argument takes on values between 0 and 2π each of these functions has the graph shown in figure 6–20.* These are the basic sine and cosine cycles discussed earlier. Observe that the basic sine cycle is 0 at its beginning, middle, and end points, while the cosine function starts at 1, ends at 1, and is -1 at its midpoint.

Now consider what the graph of $y = \sin\left(x - \frac{\pi}{4}\right)$ should look like. We know that one basic cycle of the sine function is produced as the argument goes from 0 to 2π. In this case, the argument is $x - \frac{\pi}{4}$. We find the values of x for which the argument goes from 0 to 2π as follows:

$$0 \le x - \frac{\pi}{4} \le 2\pi$$

$$\frac{\pi}{4} \le x \le 2\pi + \frac{\pi}{4} \qquad \text{Add } \frac{\pi}{4} \text{ to each member}$$

$$\frac{\pi}{4} \le x \le \frac{9\pi}{4}$$

Thus, as x takes on the values $\frac{\pi}{4}$ through $\frac{9\pi}{4}$, $x - \frac{\pi}{4}$ takes on the values from 0 through 2π, tracing out the graph of the basic sine function. Thus, the basic sine function starts at $\frac{\pi}{4}$ and ends at $\frac{9\pi}{4}$. See figure 6–21, where we mark the axis in units of $\frac{\pi}{4}$ for convenience.

The distance between the points at $\frac{9\pi}{4}$ and $\frac{\pi}{4}$ is $\frac{9\pi}{4} - \frac{\pi}{4} = \frac{8\pi}{4} = 2\pi$. This is the period of this function. The value $\frac{\pi}{4}$ is called the phase shift of the function. It is the value to which the starting point of the basic cycle shifted.

[Y=] [SIN] [(] [X|T]
[−] [π] [÷] 4 [)]
[RANGE −5.5,13.5,−2,2]

Xscl=0.7854 $\approx \frac{\pi}{4}$

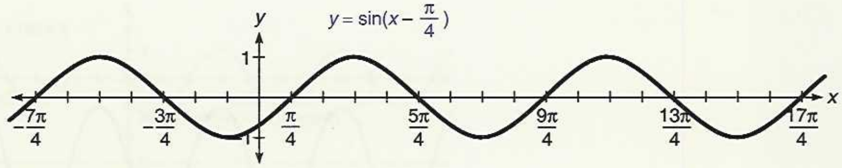


Figure 6–21

This method illustrates how to graph sine and cosine functions where the argument is of the form $Bx + C$. In these cases, we get horizontal scaling (the period) and horizontal shifts (the phase shift).

> **Algebraic methods to graph sine and cosine functions of the form**
>
> $y = A\sin(Bx + C) + D$ and
> $y = A\cos(Bx + C) + D$, where $B > 0$
>
> 1. Solve $0 \le Bx + C \le 2\pi$ so x is the middle member.
>   - This gives the left and right end points for one basic cycle.
>   - The left end point is the phase shift.
>   - The difference between the end points is the period.
> 2. The amplitude is $|A|$.
>   - This is the height of the basic graph above and below the x-axis.
>   - The graph is reflected across the horizontal axis if $A < 0$.
>
>   Draw one basic cycle with the information from steps 1 and 2.
> 3. Repeat the cycle obtained from steps 1, 2 to obtain more of the graph.
> 4. Shift the graph vertically D units.

Period and phase shift—definition and method of computation

We formalize the terms period and phase shift in the following way. We perform step 1: $0 \le Bx + C \le 2\pi$

$$-C \le Bx \le 2\pi - C \qquad \text{Subtract } C \text{ from each member}$$

$$-\frac{C}{B} \le x \le \frac{2\pi - C}{B} \qquad \text{Divide each member by } B$$

The expression $-\frac{C}{B}$ is called the **phase shift,** of the particular sine or cosine function. The difference between the left and right end points of the basic cycle, $\frac{2\pi - C}{B} - \left(-\frac{C}{B}\right) = \frac{2\pi}{B}$ is called the **period** of the function. B is also the number of complete cycles in 2π units.

To compute the phase shift and period, either memorize the definitions $-\frac{C}{B}$ and $\frac{2\pi}{B}$ or solve the inequality $0 \le Bx + C \le 2\pi$ for x. The second method is used in example 6–2 C.

■ ***Example 6–2 C***

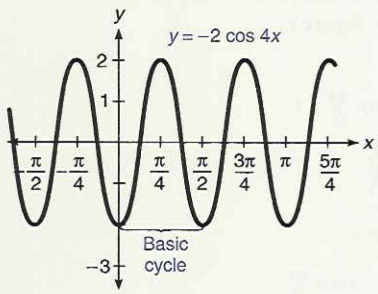
Graph each function. Show at least two cycles. State the amplitude, period, and phase shift.

1. $y = -2\cos 4x$

Step 1: $0 \le 4x \le 2\pi$

$0 \le x \le \dfrac{\pi}{2}$ Divide each member by 4

Thus one basic cycle of the cosine function starts at 0 and ends at $\dfrac{\pi}{2}$. The period is $\dfrac{\pi}{2} - 0 = \dfrac{\pi}{2}$, and the phase shift is 0.



Step 2: The amplitude is $|-2| = 2$. There is a reflection about the horizontal axis. Use this information to draw one basic cycle; this is shown in the figure.

Step 3: Draw additional cycles by repeating this pattern to the right and left of the original basic cycle. Since there is no value for D (it is therefore 0), we do not need to consider step 4.

Y=	(−) 2	COS	(4	X|T	)	RANGE −2,4,−3,3

Xscl = 0.7854

Note $\dfrac{\pi}{4} \approx 0.7854$.

2. $y = \sin\left(2x + \dfrac{\pi}{3}\right)$

Step 1: $0 \le 2x + \dfrac{\pi}{3} \le 2\pi$

$0 \le 6x + \pi \le 6\pi$ Multiply each term of each member by 3

$-\pi \le 6x \le 5\pi$ Subtract π from each member

$-\dfrac{\pi}{6} \le x \le \dfrac{5\pi}{6}$ A basic cycle starts at $-\dfrac{\pi}{6}$ and ends at $\dfrac{5\pi}{6}$

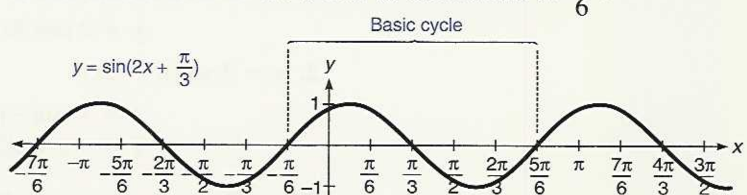
The phase shift is $-\dfrac{\pi}{6}$, and period is $\dfrac{5\pi}{6} - \left(-\dfrac{\pi}{6}\right) = \pi$.

Step 2: The amplitude is 1. We draw one basic cycle (see the figure). It is convenient to mark the x-axis in increments of $\dfrac{\pi}{6}$.

Y=	SIN	(2	X|T
+	π	÷ 3	)

RANGE −3.7,4.7,−2,2

Xscl = .5236 ≈ $\dfrac{\pi}{6}$



Step 3: We sketch one complete cycle to the left of the basic cycle and a portion of a third cycle to its right.

3. $y = 3\cos\dfrac{\pi x}{2} + 1$

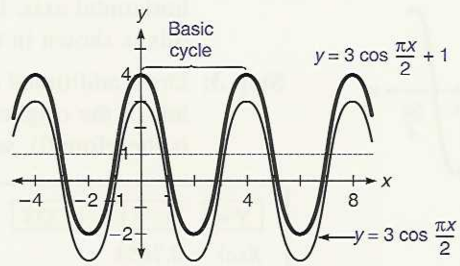
Step 1: $0 \le \dfrac{\pi x}{2} \le 2\pi$

$0 \le \pi x \le 4\pi$ Multiply each member by 2

$0 \le x \le 4$ Divide each member by π

Thus one basic cosine cycle begins at 0 and ends at 4. The phase shift is 0, and the period is $4 - 0 = 4$.

Step 2: The amplitude is 3. We draw one basic cosine cycle between 0 and 4, with amplitude 3 (see the figure).



Step 3: We repeat one cycle to the left and one to the right of the basic cycle.

Step 4: We shift the graph up 1 unit. The horizontal line $y = 1$ helps orient the figure.

Y= 3 COS (π X|T ÷ 2) + 1
RANGE −4,8,−3,5

Negative coefficients of *x*

If the coefficient of x, B, is negative in the argument $Bx + C$ we use the odd and even properties to get an equivalent expression with B positive. We can then graph the equivalent expression where the coefficient of x is positive.

■ ***Example 6–2 D***

Rewrite each function so the coefficient of x is positive.

1. $y = 2\cos(-3x)$

$y = 2\cos 3x$ $\cos\theta = \cos(-\theta)$

2. $y = 3\sin(\pi - 2x)$

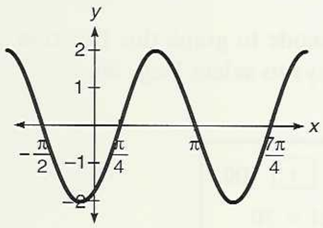
$y = 3\sin[-(2x - \pi)]$ $\pi - 2x = -(2x - \pi)$

$y = -3\sin(2x - \pi)$ $\sin(-\theta) = -\sin\theta$

Finding an equation from its properties

There are times when we will want to find the equation of a function, given some of its properties.

Example 6–2 E



Find the equation of the function in the figure, assuming it is of the form $y = A\sin(Bx + C) + D$.

Since the distance between the high and low points of the graph is 4 we know that A is 2. There is no vertical shift, so D is 0.

One cycle of the sine function begins at $\frac{\pi}{4}$ and ends at $\frac{7\pi}{4}$. We work backward from this information. We want to obtain end points of 0 and 2π:

$\frac{\pi}{4} \le x \le \frac{7\pi}{4}$ One basic cycle is between these end points

$\pi \le 4x \le 7\pi$ Multiply each term by 4 to clear fractions

$0 \le 4x - \pi \le 6\pi$ Subtract π from each member; this makes the left member 0

$\frac{0}{3} \le \frac{4x - \pi}{3} \le \frac{6\pi}{3}$ Divide each member by 3; this makes the right member 2π

$0 \le \frac{4x}{3} - \frac{\pi}{3} \le 2\pi$ $Bx + C$ is $\frac{4x}{3} - \frac{\pi}{3}$

Thus, the function is $y = 2\sin\left(\frac{4x}{3} - \frac{\pi}{3}\right)$.

One of the places trigonometry finds application is in alternating current theory in electronics. Many such applications express x in degrees instead of radians. Our graphing procedures are the same except that our limits for the basic cycle are 0° and 360°.

Example 6–2 F

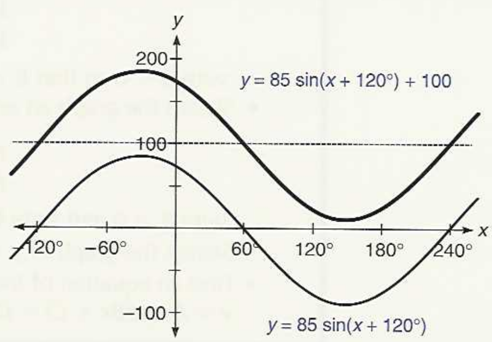
1. An AC (alternating current) signal with peak-to-peak voltage of 170 volts and phase shift of $-120°$, riding on a DC (direct current) level of 100 volts, could be described by the function $y = 85\sin(x + 120°) + 100$, where y represents volts and x is in degrees. Graph one cycle of this function.

Step 1: $0° \le x + 120° \le 360°$

$-120° \le x \le 240°$

Phase shift is $-120°$ and period is $240° - (-120°) = 360°$. We have one basic sine cycle between $-120°$ and $240°$.

Step 2: The amplitude is 85. The basic cycle is shown in the figure.



Step 3: We shift the basic cycle upward by 100 units to obtain the final graph.

A graphing calculator must be in degree mode to graph this function. Use the $\boxed{\text{MODE}}$ key, then use the cursor keys to select Deg, then select this mode with the $\boxed{\text{ENTER}}$ key.

$\boxed{\text{Y=}}$ 85 $\boxed{\text{SIN}}$ $\boxed{(}$ $\boxed{\text{X}\mid\text{T}}$ $\boxed{+}$ 120 $\boxed{)}$ $\boxed{+}$ 100
$\boxed{\text{RANGE } -140{,}260{,}-100{,}200}$ Xscl = 60, Yscl = 50

2. An electronic signal is to be modeled with the sine function. The peak-to-peak voltage is 340 volts (amplitude is 170 volts). There is a phase shift of 30°, and the period is 120°. The signal is at 200 volts above ground potential (there is a vertical shift of 200). Find the sine function that will model this signal.

The required function is of the form $y = A\sin(Bx + C) + D$. A (amplitude) is 170, and D (vertical shift) is 200.

To find B and C we proceed "backward." We know that we get one basic cycle as x varies between 30° (phase shift) and 30° + 120° (phase shift and period).

$$30° \le x \le 30° + 120°$$ We want $0° \le Bx + C \le 360°$

$$0° \le x - 30° \le 120°$$ Subtract 30° from each member so the left member is 0°

$$0° \le 3x - 90° \le 360°$$ Multiply each member by 3 so the right member is 360°

Thus, the function we want is $y = 170\sin(3x - 90°) + 200$. ■

Mastery points

Can you

- State whether each function, sine, cosine, and tangent is even or odd?
- State the domain and range of the sine, cosine, and tangent functions?
- Use the odd/even properties to compute the values of $\sin x$, $\cos x$, and $\tan x$ for negative values of x?
- Transform equations of the form

$$y = A\sin(Bx + C) + D \text{ or}$$
$$y = A\cos(Bx + C) + D$$

with $B < 0$ so that $B > 0$, using the odd/even properties?
- Sketch the graph an equation of the form

$$y = A\sin(Bx + C) + D \text{ or}$$
$$y = A\cos(Bx + C) + D$$

with $B > 0$ and state the amplitude, period, and phase shift?
- Sketch the graph of $y = \tan x$?
- Find an equation of the form $y = A\sin(Bx + C) + D$ or $y = A\cos(Bx + C) + D$, given certain required properties?

Exercise 6–2

1. From memory sketch the graphs of $y = \sin x$, $y = \cos x$, $y = \tan x$.
2. From memory, or using their graphs as an aid, state the domain, range, and period of each of the functions: sine, cosine, and tangent.
3. Using the graph of $y = \sin x$ as a guide, describe all values of x for which $\sin x$ is **a.** 1 **b.** -1 **c.** 0
4. Using the graph of $y = \cos x$ as a guide, describe all values of x for which $\cos x$ is **a.** 1 **b.** -1 **c.** 0
5. Using the graph of $y = \tan x$ as a guide, describe all values of x for which $\tan x$ is 0.
6. Use the graph of $y = \sin x$, as well as the unit circle, to describe all points for which $\sin x$ is
   a. $\frac{1}{2}$ **b.** $-\frac{\sqrt{3}}{2}$ **c.** $-\frac{\sqrt{2}}{2}$
7. Use the graph of $y = \cos x$, as well as the unit circle, to describe all points for which $\cos x$ is
   a. $\frac{1}{2}$ **b.** $-\frac{\sqrt{3}}{2}$ **c.** $-\frac{\sqrt{2}}{2}$

Use the appropriate property, odd or even, to simplify the computation of the exact value of the (a) sine, (b) cosine, and (c) tangent functions for the following values.

8. $-\frac{\pi}{3}$ **9.** $-\frac{\pi}{6}$ **10.** $-45°$ **11.** $-\frac{2\pi}{3}$

Graph three cycles of the following functions.

12. $y = 5 \sin x$ **13.** $y = 5 \cos x$ **14.** $y = \frac{2}{3} \cos x$ **15.** $y = \frac{1}{5} \sin x$
16. $y = -4 \cos x$ **17.** $y = -2 \sin x$ **18.** $y = -\frac{1}{3} \sin x$ **19.** $y = -\frac{5}{2} \sin x$
20. $y = 2 \sin x + 1$ **21.** $y = 3 \cos x - 2$ **22.** $y = -\frac{3}{4} \cos x - 2$ **23.** $y = -\frac{1}{2} \sin x + 3$

Graph three cycles of the following functions. State the amplitude, period, phase shift, and any vertical shift of each.

24. $y = 2 \sin 4x$ **25.** $y = 3 \cos \frac{x}{2}$ **26.** $y = \cos\left(x - \frac{\pi}{2}\right)$ **27.** $y = 3 \sin(2x + \pi)$
28. $y = \frac{2}{3} \sin(3x + \pi)$ **29.** $y = \frac{5}{8} \cos 5x$ **30.** $y = -\cos 3x$ **31.** $y = -\sin x$
32. $y = -\cos\left(2x + \frac{\pi}{2}\right)$ **33.** $y = -\sin\left(3x - \frac{\pi}{3}\right)$ **34.** $y = \sin(3x + \pi)$ **35.** $y = \cos(2x - \pi)$
36. $y = \cos 2\pi x$ **37.** $y = \sin \pi x$ **38.** $y = 2 \sin 3x + 2$ **39.** $y = 3 \cos 2x - 3$
40. $y = -3 \cos x + 1$ **41.** $y = -\sin 4x + 1$ **42.** $y = 2 \sin(2x - \pi) + 1$ **43.** $y = 3 \sin(3x + \pi) - 3$
44. $y = \sin \pi x + 1$ **45.** $y = 2 \cos \frac{\pi x}{2} - 2$

Use the odd/even properties of the sine and cosine functions to rewrite each of the following functions as an equivalent function in which the coefficient of x is positive.

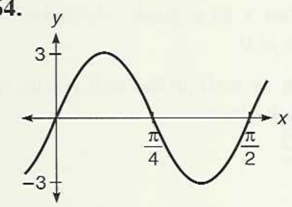
46. $y = \sin(-2x)$ **47.** $y = \cos(-x)$ **48.** $y = -\cos(-3x)$ **49.** $y = -\sin(-5x)$
50. $y = \sin(-x - 3)$ **51.** $y = \cos(-2x + 4)$ **52.** $y = \sin(-x) - 3$ **53.** $y = \cos(-2x) + 4$
54. $y = -3 \cos\left(-2x + \frac{\pi}{2}\right)$ **55.** $y = 2 \sin\left(-\frac{x}{3} - \pi\right)$

Use the odd/even properties of the sine and cosine functions to rewrite each of the following functions as an equivalent function in which the coefficient of x is positive. Then graph one cycle of the function.

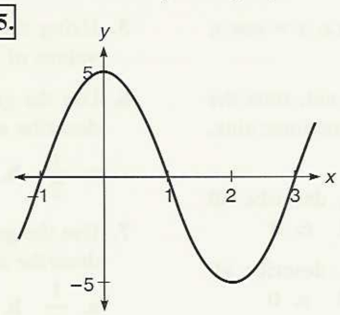
56. $y = \sin(-x)$ **57.** $y = \cos(-2x)$ **58.** $y = \cos\left(-x - \frac{\pi}{3}\right)$ **59.** $y = 2 \sin(-2x + \pi)$
60. $y = -\sin(-2\pi x + \pi)$ **61.** $y = -\cos(-\pi x)$ **62.** $y = \sin(-\pi x + 1)$ **63.** $y = 2 \cos(-3\pi x - 2)$

Assume that each of the following graphs is the graph of a sine function of the form $y = A\sin(Bx + C)$. Find values of A, B, C, and D that would produce each graph, and write the corresponding equation using these values.

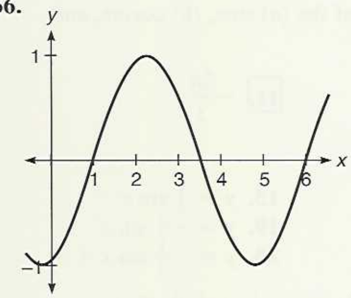
64.



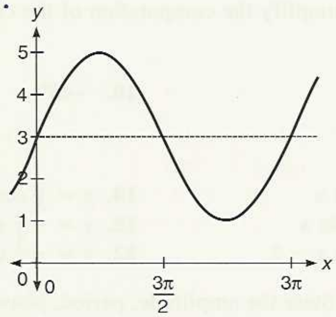
65.



66.



67.



68. Do problem 64, assuming that the graph is a cosine function of the form $y = A\cos(Bx + C) + D$.

69. Do problem 65, assuming that the graph is a cosine function of the form $y = A\cos(Bx + C) + D$.

70. Do problem 66, assuming that the graph is a cosine function of the form $y = A\cos(Bx + C) + D$.

71. Do problem 67, assuming that the graph is a cosine function of the form $y = A\cos(Bx + C) + D$.

Graph one cycle of each of the following functions. Mark the horizontal axis in degrees.

72. $y = 3\sin(x + 60°)$ **73.** $y = -50\cos(x - 120°)$

74. $y = 25\cos 3x$ **75.** $y = 10\sin(2x - 180°)$

76. An electronic signal modeled with the sine function has a peak-to-peak voltage of 120 volts (amplitude is 60 volts), phase shift of 90°, and period of 54°. Find an equation of the sine function that will model this signal.

77. An ocean wave is being modeled with the sine function. Its amplitude is 6 feet and its phase shift (with respect to another wave) is −180°. If the period is 720°, find an equation of the sine function that will model this wave.

78. One of the components of a function that could describe the earth's ice ages for the last 500,000 years is described by a sine function with amplitude 0.5, period $\frac{360}{43}$°, 0° phase shift, and vertical translation 23.5. Find an equation for this component.

79. The activity of sunspots seems to follow an 11-year cycle. Assuming that this activity can be roughly modeled with a sine wave, construct a sine function with period $\frac{360}{11}$°, amplitude 1, phase shift 90°, and vertical translation 2.

80. Graph the following functions on the same set of axes.

a. $y = \sin x$ **b.** $y = \sin 3x$ **c.** $y = \sin\frac{x}{3}$

81. Graph the following functions on the same set of axes:

a. $y = \sin x$ **b.** $y = \sin\left(x + \frac{\pi}{2}\right)$

c. $y = \sin x + \frac{\pi}{2}$

82. Graph the following functions on the same set of axes:

a. $y = \cos\left(\frac{\pi}{2} - x\right)$ **b.** $y = \sin x$

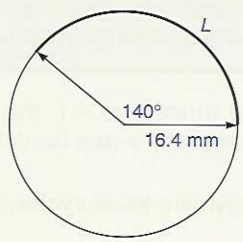
Draw a conclusion from the graph. *Hint:* Rewrite part a as $y = \cos\left(-x + \frac{\pi}{2}\right)$.

Test the function for the even/odd property. State which type of symmetry the graph would have based upon being even, odd, or neither even nor odd.

83. $f(x) = x$
84. $f(x) = 3x$
85. $f(x) = 3x^2$
86. $f(x) = -x^2$
87. $f(x) = 2x^4 - 4x^2$
88. $f(x) = 2x^3 - 4x$
89. $f(x) = 3 \sin x$
90. $f(x) = \cos x$
91. $f(x) = \sin^2 x$
92. $f(x) = x^2 \tan x$
$\boxed{93.}$ $f(x) = \dfrac{\cos x}{x}$
94. $f(x) = \sin x + \cos x$
95. $f(x) = (\sin x)(\cos x)$
96. $f(x) = \dfrac{\sin x}{x}$
97. $f(x) = \sin^3 x$

Skill and review

1. Find the degree measure of an angle of $\frac{5\pi}{8}$ radians.
2. Find the radian measure of an angle of measure 345°.
3. Find L in the figure, to the nearest 0.1 millimeter.



4. θ is an angle in standard position, $\sin \theta = \frac{\sqrt{3}}{5}$, and θ terminates in quadrant II. If the point (a,b) is on the terminal side of θ, and $b = -3$, find a.
5. Solve the right triangle ABC if the measure of angle A is 30° and the length of side c is 12. Leave the answers in exact form.

6–3 The tangent, cotangent, secant, and cosecant functions

The function $y = \cot 2x$ arises in chaos theory, a branch of mathematics that has developed only recently. Chaos theory is used to model things like population growth. Graph three cycles of this function.

The tangent and cotangent functions

The tangent and cotangent functions are π-periodic, so the basic cycle for each is π units "wide" instead of 2π units, as with the sine and cosine functions. Figure 6–22 shows a basic cycle for the tangent and cotangent functions. The basic graph of the cotangent function can be obtained by plotting points or with a graphics calculator.

The basic tangent cycle starts at $-\frac{\pi}{2}$ and ends at $\frac{\pi}{2}$. The basic cotangent cycle starts at 0 and ends at π.

Graphing functions of the form

$$y = A \tan(Bx + C) \quad \text{and}$$
$$y = A \cot(Bx + C)$$

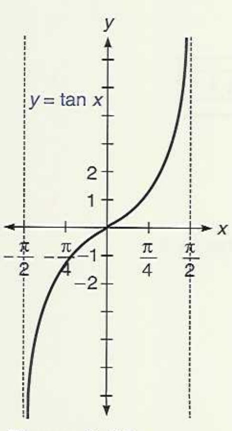


Figure 6–22a

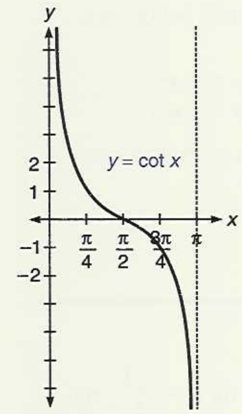


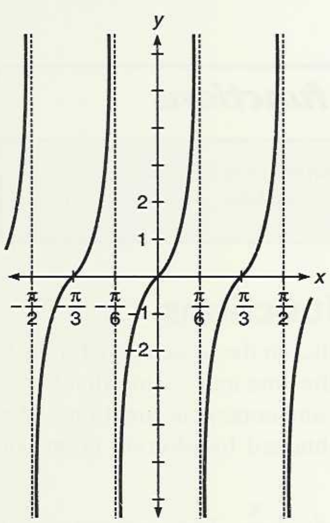
Figure 6–22b

is done in a manner similar to that for the sine and cosine functions. The values of these functions gets arbitrarily large in absolute value, so the concept of amplitude is not especially useful. We consider only cases where $|A| = 1$. We also do not stress phase shift or vertical shift for these functions.

> **To graph functions of the form**
>
> $y = \tan(Bx + C)$ and
>
> $y = \cot(Bx + C)$, $B > 0$
>
> using algebraic methods
>
> - For the tangent function, solve $-\frac{\pi}{2} < Bx + C < \frac{\pi}{2}$ for x; for the cotangent function solve $0 < Bx + C < \pi$ for x.
>   This gives the left and right end points for one basic cycle. The difference between the end points is the period.
> - Use the values from the first step to draw one basic cycle. Repeat this cycle to obtain as much of the graph as desired.

Note If the coefficient of the function is −1, the graph is obtained as stated, but is then reflected across the x-axis (as with any other function).

■ *Example 6–3 A*



Graph the function. Show at least two basic cycles. State the period.

1. $y = \tan 3x$

Step 1: $-\frac{\pi}{2} < 3x < \frac{\pi}{2}$ Multiply each member by $\frac{1}{3}$

$-\frac{\pi}{6} < x < \frac{\pi}{6}$

The period is $\frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$.

Step 2: We draw a basic cycle between $-\frac{\pi}{6}$ and $\frac{\pi}{6}$, then repeat two cycles, one on each side of the basic cycle.

Y=	TAN	(	3	X|T	)	RANGE −2,2,−4,4

2. $y = \cot\left(2x - \frac{\pi}{3}\right)$

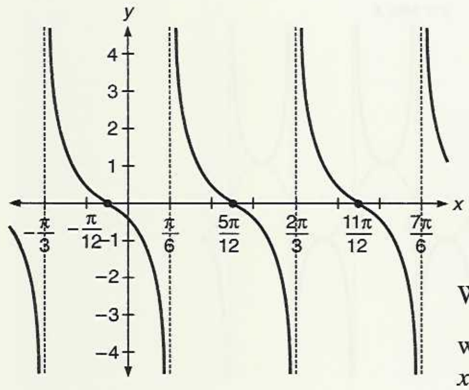
Step 1: $0 < 2x - \frac{\pi}{3} < \pi$ Multiply each member by 3

$0 < 6x - \pi < 3\pi$

$\pi < 6x < 4\pi$

$\frac{\pi}{6} < x < \frac{2\pi}{3}$

The period is $\frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{2}$.



Step 2: Draw a basic cotangent cycle from $\frac{\pi}{6}$ to $\frac{2\pi}{3}$. Repeat this cycle on either side to obtain the graph shown in the figure.

$\boxed{\text{Y=}}$ $\boxed{(}$ $\boxed{\text{TAN}}$ $\boxed{(}$ 2 $\boxed{\text{X}\mid\text{T}}$ $\boxed{-}$ $\boxed{\pi}$ $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{)}$ $\boxed{x^{-1}}$
$\boxed{\text{RANGE } -1.5,4,-4,4}$ Xscl = 0.5236

We already know that the tangent function is odd. Since $\cot x = \frac{1}{\tan x}$, except where $\tan x = 0$, cotangent is also an odd function. Thus, if the coefficient of x is negative we use this fact, as we did for the sine function.

■ ***Example 6–3 B***

1. Rewrite $y = \tan(\pi - x)$ so the coefficient of x is positive.

$y = \tan(\pi - x)$

$y = \tan[-(x - \pi)]$ $\quad \pi - x = -(x - \pi)$

$y = -\tan(x - \pi)$ $\quad \tan(-x) = -\tan x$, since tangent is an odd function.

2. Determine if the function $f(x) = x \cot x$ is even, odd, or neither.

$$\begin{aligned} f(-x) &= (-x)\cot(-x) \\ &= (-x)(-\cot x) \quad && \cot(-x) = -\cot(x) \\ &= x \cot x \\ &= f(x) \end{aligned}$$

Since $f(-x) = f(x)$, this is an even function.

The secant and cosecant functions

To graph variations of the secant and cosecant functions we use the fact that they are reciprocals of the cosine and sine functions, respectively. Figures 6–23 and 6–24 show the graphs of the cosecant and secant functions. Observe that the ranges of both functions are $|y| \geq 1$, and that they have vertical asymptotes where their reciprocal function (cosine or sine) is zero. The domain of each function is the set of real numbers excluding the points at which they have vertical asymptotes.

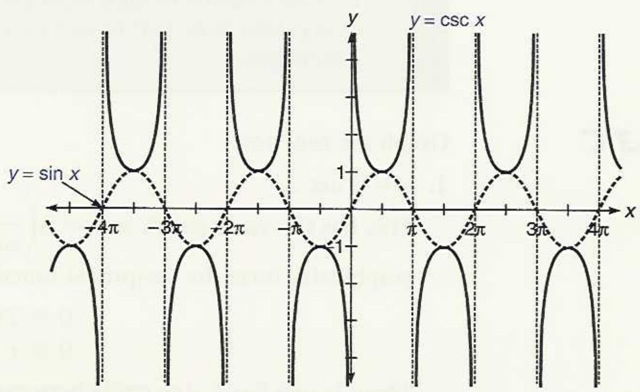


Figure 6–23

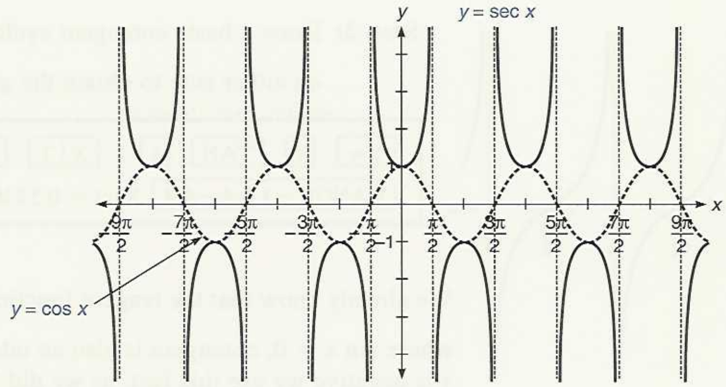


Figure 6–24

To understand the graph of $y = \csc x$, observe that $\csc x = \dfrac{1}{\sin x}$. Where $\sin x = 1$, $\csc x = 1$. Where $\sin x = -1$, $\csc x = -1$. Where $|\sin x| < 1$, $|\csc x| > 1$. As $|\sin x|$ gets closer to zero, $\left|\dfrac{1}{\sin x}\right|$ gets larger and larger. This is where the vertical asymptotes occur. Observe in the figures that the graph of the cosecant and secant functions touch the graphs of the sine and cosine functions, respectively, at their highest points, and that they then increase in absolute value from that point toward their vertical asymptotes. We use the sine and cosine functions to help us graph the cosecant and secant functions.

To graph $y = A\csc(Bx + C)$ or $y = A\sec(Bx + C)$ using algebraic methods

- Graph the appropriate reciprocal function,

$$y = A\sin(Bx + C) \quad \text{or} \quad y = A\cos(Bx + C)$$

- Sketch in vertical asymptotes wherever the sine or cosine function is zero.
- Create the cosecant or secant graph by starting at the highest and lowest points of the sine or cosine graph and sketching values that increase in absolute value from that point as x approaches the vertical asymptotes. Note that these functions are not defined at the asymptotes.

■ *Example 6–3 C*

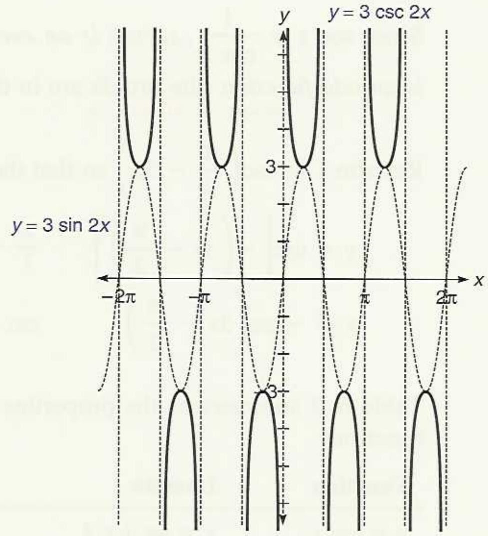
Graph the function.

1. $y = 3\csc 2x$

This has the same graph as $y = 3\left(\dfrac{1}{\sin 2x}\right)$. We graph $y = 3\sin 2x$, then graphically form the reciprocal function, as shown in the figure.

$$0 \le 2x \le 2\pi$$
$$0 \le x \le \pi$$

There is one basic sine cycle between 0 and π. The resultant graph is shown in the figure.



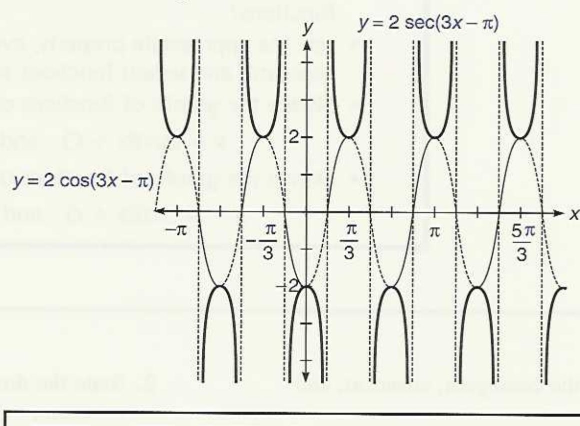
Y= 3 ÷ SIN (2 X|T) RANGE −6.5,6.5,−6,6
Xscl = 1.5708

2. $y = 2\sec(3x - \pi)$

We graph $y = 2\cos(3x - \pi)$.

$$0 \le 3x - \pi \le 2\pi$$
$$\pi \le 3x \le 3\pi$$
$$\frac{\pi}{3} \le x \le \pi$$

There is one basic cosine cycle from $\frac{\pi}{3}$ to π. The resultant graph is shown in the figure.



Y= 2 ÷ COS (3 X|T − π)
RANGE −6,12,−4,4 Xscl = 1.0472

Since $\sec x = \dfrac{1}{\cos x}$, *secant is an even function.* Csc $x = \dfrac{1}{\sin x}$, so *cosecant is an odd function* (the proofs are in the exercises).

■ ***Example 6–3 D***

Rewrite $y = \csc\left(\dfrac{\pi}{2} - 3x\right)$ so that the coefficient of x is positive.

$$y = \csc\left[-\left(3x - \frac{\pi}{2}\right)\right] \qquad \frac{\pi}{2} - 3x = -\left(3x - \frac{\pi}{2}\right)$$

$$y = -\csc\left(3x - \frac{\pi}{2}\right) \qquad \csc(-\theta) = \csc\theta\text{; cosecant is an odd function}$$

■

Table 6–3 summarizes the properties of the cotangent, secant, and cosecant functions.

Function	Domain	Range	Period
$y = \csc x$	$x \neq k\pi, k \varepsilon J$	$\lvert y\rvert \geq 1$	2π
$y = \sec x$	$x \neq \dfrac{\pi}{2} + k\pi, k \varepsilon J$	$\lvert y\rvert \geq 1$	2π
$y = \cot x$	$x \neq k\pi, k \varepsilon J$	R	π

$\csc(-x) = -\csc x$ (odd, origin symmetry)
$\sec(-x) = \sec x$ (even, y-axis symmetry)
$\cot(-x) = -\cot x$ (odd, origin symmetry)

Table 6–3

Mastery points

Can you

- Sketch the graphs of the cotangent, cosecant, and secant functions?
- State the domain and range of the cotangent, cosecant, and secant functions?
- Use the appropriate property, even or odd, to transform cotangent, cosecant, and secant functions so the argument is positive?
- Sketch the graphs of functions of the form

  $y = \tan(Bx + C)$ and $y = \cot(Bx + C)$, $B > 0$?
- Sketch the graphs of functions of the form

  $y = A\csc(Bx + c)$ and $y = A\sec(Bx + C)$, $B > 0$?

Exercise 6–3

1. Sketch the graphs of the cotangent, cosecant, and secant functions.
2. State the domain, range, and period for the cotangent, cosecant, and secant functions.
3. Use the identity $\csc x = \dfrac{1}{\sin x}$ to show that cosecant is an odd function.
4. Use the identity $\sec x = \dfrac{1}{\cos x}$ to show that secant is an odd function.

5. Use the identity $\cot x = \dfrac{1}{\tan x}$ to show that cotangent is an odd function.

6. Show that the function $f(x) = \sin^2 x \cos x$ is an even function.

7. Show that the function $f(x) = \sin^2 x \tan x$ is an odd function.

8. Show that the function $f(x) = \sin x \tan x$ is an even function.

Graph three cycles of the following functions.

9. $y = 2 \tan x$ **10.** $y = -\cot x$ **11.** $y = \tan \dfrac{x}{4}$ **12.** $y = \cot \dfrac{x}{2}$

13. $y = \cot\left(x - \dfrac{\pi}{2}\right)$ **14.** $y = 3 \tan(2x + \pi)$ **15.** $y = -\cot\left(2x + \dfrac{\pi}{2}\right)$ **16.** $y = -\tan\left(3x - \dfrac{\pi}{3}\right)$

17. $y = \cot 2\pi x$ **18.** $y = \tan \pi x$

Use the odd/even properties of the tangent and cotangent functions to rewrite each of the following functions as an equivalent function in which the coefficient of x is positive. Then graph one cycle of the function.

19. $y = \tan(-2x)$ **20.** $y = \cot(-x)$ **21.** $y = -\cot(-\pi x)$

22. $y = -\tan(-2\pi x)$ **23.** $y = \tan(-x - \pi)$ **24.** $y = \cot(-2x + \pi)$

Graph three cycles of the following functions.

25. $y = \frac{2}{3} \csc x$ **26.** $y = \frac{1}{5} \sec x$ **27.** $y = -4 \csc x$ **28.** $y = 2 \sec 4x$

29. $y = 3 \csc \dfrac{x}{2}$ **30.** $y = \csc\left(x - \dfrac{\pi}{2}\right)$ **31.** $y = 3 \sec(2x + \pi)$ **32.** $y = \frac{2}{3} \sec(3x + \pi)$

33. $y = \csc\left(\dfrac{x}{2} - \dfrac{\pi}{3}\right)$ **34.** $y = \csc 2\pi x$ **35.** $y = \sec \pi x$

Use the odd/even properties of the sine and cosine functions to graph at least one cycle of each of the following functions.

36. $y = \sec(-2x)$ **37.** $y = \csc(-x)$ **38.** $y = 3 \csc\left(-2x + \dfrac{\pi}{2}\right)$ **39.** $y = 2 \sec\left(-\dfrac{x}{3} - \dfrac{\pi}{2}\right)$

40. Chaos theory is a recent development in mathematics. It finds application in modeling things like population growth. Gilbert Strang of the Massachusetts Institute of Technology has shown[3] that the function $y = \dfrac{1}{2}\left(\cot x - \dfrac{1}{\cot x}\right)$ arises in chaos theory. It can be shown (chapter 7) that this is equivalent to the function $y = \cot 2x$. Graph three cycles of this function.

[3] "A Chaotic Search for i," by Gilbert Strang, *The College Mathematics Journal*, Vol. 22, No. 1, January 1991.

Skill and review

1. Graph the function $f(x) = x^4 - x^3 - 7x^2 + x + 6$. Recall that the zeros of the right member are the x-intercepts, and that the rational zero theorem and synthetic divison can be used to help find these zeros.

2. Graph the quadratic function $f(x) = x^2 + 6x - 4$. Find the vertex by completing the square and putting the equation in vertex form $f(x) = a(x - h)^2 + k$.

3. Rationalize the denominator of $\dfrac{\sqrt{3}}{\sqrt{3} - 6}$.

4. Multiply the complex numbers $(3 - 7i)(2 + 3i)$.

5. Test the function $f(x) = x^2 - \cos x$ for the even or odd property. State which type of symmetry the graph of this function would exhibit.

6. Graph the function $f(x) = 3 \sin x$.

7. Graph the function $f(x) = -\cos 2x$.

8. Graph the function $f(x) = 2 \sin\left(x + \dfrac{\pi}{5}\right)$.

6–4 The inverse sine, cosine, and tangent functions

Firefighters use the formula $d = \frac{h}{4} + 2$ to obtain the safe height of a fire ladder resting against a wall, where h is the height up the wall and d is the distance away from the wall at which the ladder should rest. Let θ represent the angle the ladder makes with the ground. Write the value of θ in terms of the variable h.

To solve problems like this we need the functions presented in this section—the inverse trigonometric functions. We have already used them when we solved equations like $\sin \theta = \frac{1}{2}$, where we stated that $\theta' = \sin^{-1}\frac{1}{2}$ and that θ' is 30° or $\frac{\pi}{6}$ (throughout chapter 5).

We know (section 4–5) that the inverse of a function is formed by interchanging the first and second components of all of the ordered pairs in the function, and that a function has an inverse if and only if it is a one-to-one function. We also know (section 3–5) that a function is one to one if and only if it passes the horizontal line test.

An examination of the graphs of any of the trigonometric functions shows that they fail the horizontal line test and are therefore not one to one. These functions are all periodic, and no periodic function can be one to one. Nevertheless, there is a real need for inverses of the trigonometric functions.

We can define these inverse functions by limiting the domain of a given trigonometric function so that this limited part of the function (which is itself a function) is one to one and includes the entire range of the function.

The inverse sine function

Figure 6–25 shows this idea for the sine function. For the sine function, we select that portion of the domain for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. This new function is one to one and therefore has an inverse. We call this function the **inverse sine function,** sine^{-1}, abbreviated $\sin^{-1}$. The ordered pairs of this function are formed by reversing the coordinates of the ordered pairs in the restricted portion of the sine function previously discussed. The figure shows this idea for several points. For example, $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, so the ordered pair $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ is an element of the sine function. Thus, the ordered pair $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ is an element of the inverse sine function. This is also written $\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$.

In general, if an ordered pair (x,y) is in the inverse sine function, then the ordered pair (y,x) is in the sine function. Thus, if $y = \sin^{-1}x$, then $x = \sin y$. Using this idea and figure 6–25 we make the following definition.

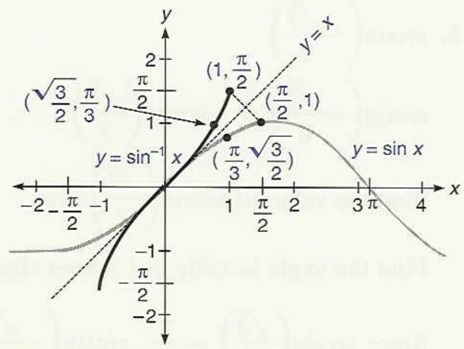
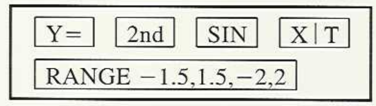


Figure 6–25



Inverse sine function

$y = \sin^{-1}x$ means

1. $\sin y = x$
2. $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
3. $-1 \le x \le 1$

Concept

$\sin^{-1}x$ is the angle in quadrant I or IV whose sine is x.

Note When we refer to quadrant IV we refer to angles with negative measure.

The domain of the sine^{-1} function is $-1 \le x \le 1$, which is the range of the sine function. The range of the sine^{-1} function is $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$, which is the domain of the one-to-one portion of the sine function that we selected above (figure 6–25).

Another notation for $\sin^{-1}x$ is **arcsin *x*.** This is because the word arc can refer to an angle, so arcsin x means ‘‘the arc (angle) whose sine is x.’’

It can be seen in its graph that *the inverse sine function is an odd function.* This can help in computations, because it means that $\sin(-x) = -\sin x$. This odd property and table 6–1 can be helpful in solving for exact values of the inverse sine function.

Although we defined the inverse trigonometric functions using radian measure we often want the result in degrees. For the sine^{-1} function this corresponds to a range of $-90° \le \sin^{-1}x \le 90°$.

■ ***Example 6–4 A***

Find the exact value both in radians and degrees.

1. $\sin^{-1}\frac{1}{2}$

Find the angle in table 6–1 whose sine is $\frac{1}{2}$. This is $\frac{\pi}{6}$, or 30°.

2. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

This uses the alternate notation for the inverse sine function

$$\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\arcsin\left(\frac{\sqrt{2}}{2}\right)$$

$\sin^{-1}$ is an odd function

Find the value of $\arcsin\left(\frac{\sqrt{2}}{2}\right)$ first.

Find the angle in table 6–1 whose sine is $\frac{\sqrt{2}}{2}$. This is $\frac{\pi}{4}$ or 45°.

Since $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$, $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$, or $-45°$. ■

For most values of x, $\sin^{-1}x$ can only be approximated using a calculator. As noted in section 5–2 most calculators use the $\boxed{\text{sin}}$ key, prefixed by another key such as $\boxed{\text{SHIFT}}$, $\boxed{\text{2nd}}$, $\boxed{\text{INV}}$, or $\boxed{\text{ARC}}$. We will assume the key is called the $\boxed{\text{SHIFT}}$ key.

■ ***Example 6–4 B***

Find $\arcsin(-0.9249)$ to the nearest 0.01 radians and 0.1°.

$$y = \arcsin(-0.9249)$$
$$y \approx -1.18 \text{ (radians) or } -67.7°$$

Use .9249 $\boxed{+/-}$ $\boxed{\text{SHIFT}}$ $\boxed{\text{sin}}$ (TI-81: $\boxed{\text{2nd}}$ $\boxed{\text{SIN}}$ $\boxed{(-)}$ .9249 $\boxed{\text{ENTER}}$) in both radian and degree modes. ■

The inverse cosine function

The inverse cosine function (cosine^{-1}) is defined in a manner similar to the inverse sine function. The one-to-one part chosen is for $0 \le x \le \pi$. See figure 6–26.

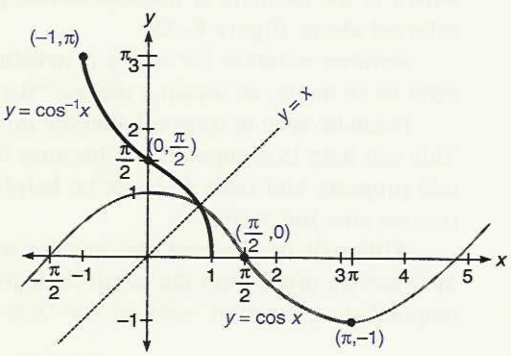


Figure 6–26

$\boxed{\text{Y=}}$ $\boxed{\text{2nd}}$ $\boxed{\text{COS}}$ $\boxed{\text{X|T}}$ $\boxed{\text{RANGE } -1.5,1.5,-.5,4}$ Xscl=1.5708

The graph of $y = \cos^{-1}x$ is the reflection of this portion of the cosine function across the line $y = x$. We can see from the graph that the domain of the cosine^{-1} function is $-1 \le x \le 1$, and the range is $0 \le \cos^{-1}x \le \pi$. The inverse cosine function is defined as follows.

Inverse cosine function

$y = \cos^{-1}x$ means

1. $\cos y = x$
2. $0 \le y \le \pi$
3. $-1 \le x \le 1$

Concept

$\cos^{-1}x$ is the angle in quadrant I or II whose cosine is x.

Arccos x also means $\cos^{-1}x$. Note that in degrees, $0° \le \cos^{-1}x \le 180°$.

The inverse cosine function is neither even nor odd. A useful fact to know, however, is the following identity.

$$\cos^{-1}(-x) = \pi - \cos^{-1}x$$

This identity, along with table 6–1, is useful in finding exact values for the inverse cosine function. Observe that in degree measure this can be written

$$\cos^{-1}(-\theta) = 180° - \cos^{-1}\theta$$

Example 6–4 C illustrates using the inverse cosine function.

■ ***Example 6–4 C***

Find the value both in radians and degrees.

1. $\cos^{-1}(-\frac{1}{2})$; find the exact value.

Find $\cos^{-1}\frac{1}{2}$ first. This is $\frac{\pi}{3}$, 60° from table 6–1.

$$\begin{aligned}\cos^{-1}(-\tfrac{1}{2}) &= \pi - \cos^{-1}\tfrac{1}{2} \text{ or } 180° - \cos^{-1}\tfrac{1}{2}\\ &= \pi - \frac{\pi}{3} \quad \text{ or } 180° - 60°\\ &= \frac{2\pi}{3} \text{ or } 120°\end{aligned}$$

2. arccos (1)

Since $\cos\theta = 1$ this is 0, or 0°.

3. $\cos^{-1}(-0.5141)$; round radians to 0.01, degrees to 0.1°.

$$y = \cos^{-1}(-0.5141)$$
$$y \approx 2.11 \text{ (radians) or } 120.9°$$

Use .5141 $\boxed{+/-}$ $\boxed{\text{SHIFT}}$ $\boxed{\cos}$ (TI-81: $\boxed{\text{2nd}}$ $\boxed{\text{COS}}$ $\boxed{(-)}$ .5141 $\boxed{\text{ENTER}}$) in radian mode, then degree mode. ■

The inverse tangent function

The inverse of the tangent function (tangent^{-1}) is based on the basic cycle of the tangent function, from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. This is one to one. The graph of $y = \tan^{-1}x$ is the reflection of the basic cycle across the line $y = x$. See figure 6–27.

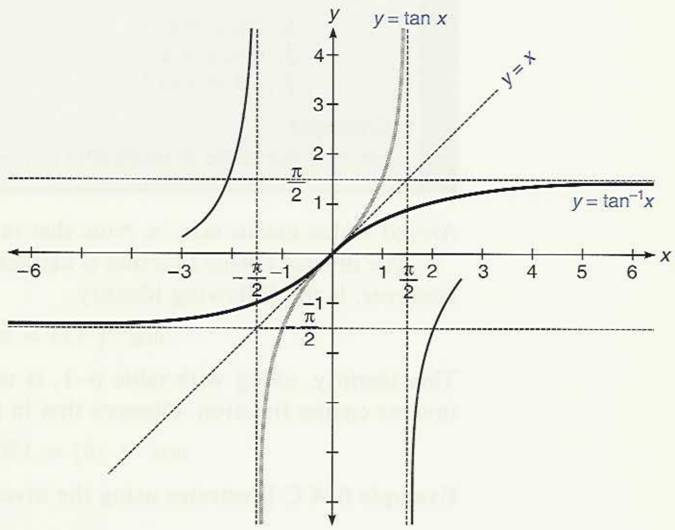


Figure 6–27

$\boxed{\text{Y=}}$ $\boxed{\text{2nd}}$ $\boxed{\text{TAN}}$ $\boxed{\text{X}\mid\text{T}}$
$\boxed{\text{RANGE } -5,5,-2,2}$
Xscl = 1.5708

The domain of the tangent^{-1} function is all the real numbers, and the range is the values strictly between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. The range of the tangent^{-1} function is in quadrants I and IV, as is the range of the sine^{-1} function, except that the points $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are not included. *Arctan x* is another notation for $\tan^{-1}x$.

Inverse tangent function

$y = \tan^{-1}x$ means

1. $\tan y = x$
2. $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Concept

$\tan^{-1}x$ is the angle in quadrant I or IV whose tangent is x.

The inverse tangent function is an odd function. Thus,

$$\tan^{-1}(-x) = -\tan^{-1}x$$

This and table 6–1 can help find exact values of this function, which is illustrated in example 6–4 D.

■ ***Example 6–4 D***

Find the required value both in degrees and radians.

1. $\tan^{-1}(\sqrt{3})$

Table 6–1 shows this is $\frac{\pi}{3}$ (radians) or 60°.

2. arctan 0.9697

$$y = \arctan 0.9697$$
$$y \approx 0.77 \text{ (radians) or } 44.1°.$$

Use .9697 $\boxed{\text{SHIFT}}$ $\boxed{\text{tan}}$ (TI-81: $\boxed{\text{2nd}}$ $\boxed{\text{TAN}}$ .9697 $\boxed{\text{ENTER}}$) both in radian and then degree modes. ■

The domains and ranges of the inverse sine, cosine, and tangent functions are summarized in table 6–4, as well as the property for dealing with $-x$. The quadrants to which the ranges correspond are also indicated.

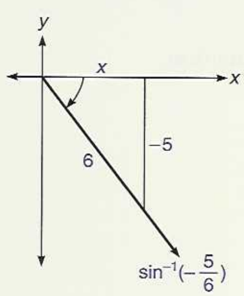
Function	Domain	Range	Quadrants
$y = \sin^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	I, IV
$y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$	I, II
$y = \tan^{-1}x$	R	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	I, IV

$\sin^{-1}(-x) = -\sin^{-1}x$ (odd)
$\cos^{-1}(-x) = \pi - \cos^{-1}x$ (neither even nor odd)
$\tan^{-1}(-x) = -\tan^{-1}x$ (odd)

Table 6–4

It is often important (especially in the study of the calculus) to simplify expressions that involve composition of the trigonometric and inverse trigonometric functions. This can often be done with the aid of a reference triangle (section 5–4), as illustrated in example 6–4 E. Remember that *any expression of the form $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$ can be interpreted to represent an angle.*

■ ***Example 6–4 E***

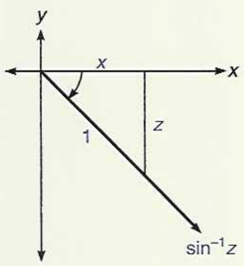


Simplify each expression.

1. $\sec[\sin^{-1}(-\frac{5}{6})]$

$\sin^{-1}(-\frac{5}{6})$ represents an angle in quadrant IV (since $-\frac{5}{6} < 0$). We construct a reference triangle for such an angle as shown. We compute x to be $\sqrt{11}$.

The cosine for this angle is $\frac{\sqrt{11}}{6}$, and the secant is the reciprocal of this value, $\frac{6}{\sqrt{11}}$ or $\frac{6\sqrt{11}}{11}$. Thus, $\sec\left[\sin^{-1}\left(-\frac{5}{6}\right)\right] = \frac{6\sqrt{11}}{11}$.



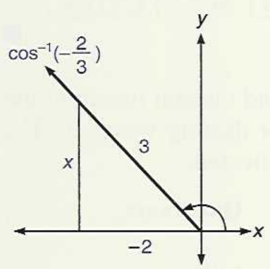
2. $\cos(\sin^{-1}z)$, $z < 0$

Since $z < 0$, $\sin^{-1}z$ is an angle in quadrant IV. The reference triangle is for an angle in quadrant IV whose sine is z, $z < 0$.

We compute $x = \sqrt{1 - z^2}$ (Pythagorean theorem).

Thus the cosine of this angle is $\frac{x}{1} = x = \sqrt{1 - z^2}$.

Thus, $\cos(\sin^{-1}z) = \sqrt{1 - z^2}$ if $z < 0$.



3. $\tan[\cos^{-1}(-\frac{2}{3})]$

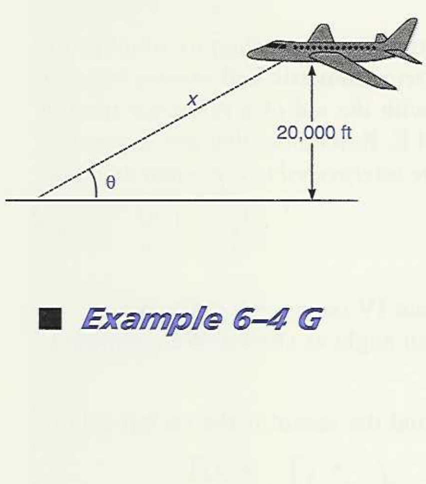
$\cos^{-1}(-\frac{2}{3})$ is an angle in quadrant II, since $-\frac{2}{3} < 0$. The figure shows a reference triangle for this angle. We compute $x = \sqrt{5}$, so the tangent of this angle is $\frac{x}{-2} = -\frac{\sqrt{5}}{2}$.

Thus, $\tan[\cos^{-1}(-\frac{2}{3})] = -\frac{\sqrt{5}}{2}$.

4. $\sin^{-1}\left(\sin\frac{7\pi}{6}\right) = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

One application of the inverse trigonometric functions is to describe an angle in a given situation by using a mathematical expression.

■ *Example 6–4 F*



A jet aircraft is flying at an altitude of 20,000 feet. If x represents the distance from a ground observer to the aircraft (also in feet), describe the angle of elevation θ from the observer to the aircraft in terms of an inverse trigonometric function.

We represent the situation as shown in the figure. We can see that $\sin\theta = \frac{20{,}000}{x}$, so $\theta = \sin^{-1}\frac{20{,}000}{x}$.

Another way in which the inverse trigonometric functions are used is to describe one of the solutions to a trigonometric equation.

■ *Example 6–4 G*

Describe one solution, using an inverse trigonometric function.

1. $\cos 2\alpha = 0.55$

$$2\alpha = \cos^{-1}0.55$$
$$\alpha = \tfrac{1}{2}\cos^{-1}0.55$$

2. $3\sin\theta = 0.69$

$$\sin\theta = 0.23$$
$$\theta = \sin^{-1}0.23$$

3. $A \sin Bx = C, A \neq 0, B \neq 0$

$$\sin Bx = \frac{C}{A}$$

$$Bx = \sin^{-1}\frac{C}{A}$$

$$x = \frac{1}{B}\sin^{-1}\frac{C}{A}$$

■

The composition of a function with its inverse always produces the same result (section 4–5), x. That is, $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ for all x in the domain of the appropriate function. Example 6–4 E (part 4) showed that $\sin^{-1}(\sin x)$ is not necessarily x. This is similar to saying $\sqrt{x^2}$ is not x for all x. (Think of a value of x for which $\sqrt{x^2} \neq x$.) The problem arises because the sine function does not have an inverse function. Sine^{-1} is the inverse of only a restricted subset of the sine function, for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Thus, $\sin^{-1}(\sin x) = x$ only if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

Taking the functions in the other direction, $\sin(\sin^{-1}x) = x$ whenever $-1 \leq x \leq 1$, because this is the domain of the sine^{-1} function. Similar reasoning allows us to make the following statements.

$$\sin^{-1}(\sin x) = x \quad \text{if and only if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$\cos^{-1}(\cos x) = x \quad \text{if and only if } 0 \leq x \leq \pi$$
$$\tan^{-1}(\tan x) = x \quad \text{if and only if } -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$\cos(\cos^{-1}x) = x \quad \text{if and only if } -1 \leq x \leq 1$$
$$\sin(\sin^{-1}x) = x \quad \text{if and only if } -1 \leq x \leq 1$$
$$\tan(\tan^{-1}x) = x \text{ for any } x$$

Mastery points

Can you

- State the domain and range of the inverse sine, cosine, and tangent functions?
- Find exact values, both in radians and degrees, for expressions of the form $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$, for appropriate values of x, using table 6–1?
- Find approximate values, both in radians and degrees, for expressions of the form $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$, using a calculator?
- Simplify expressions that involve combinations of the trigonometric and inverse trigonometric functions, using a reference triangle where appropriate?
- Use the inverse sine, cosine, and tangent functions to describe an angle in a given situation by using a mathematical expression?

Exercise 6–4

1. Sketch the graph of each function.
a. inverse sine **b.** inverse cosine **c.** inverse tangent

2. State the domain and range of each function.
a. inverse sine **b.** inverse cosine **c.** inverse tangent

Find exact values for each of the following expressions in both radians and degrees.

3. $\cos^{-1}\dfrac{\sqrt{2}}{2}$ **4.** $\sin^{-1}(-\frac{1}{2})$ **5.** $\arcsin\dfrac{\sqrt{3}}{2}$ **6.** $\arcsin\left(-\dfrac{\sqrt{2}}{2}\right)$ **7.** $\sin^{-1}1$

8. $\cos^{-1}(-\frac{1}{2})$ **9.** $\arccos\dfrac{\sqrt{3}}{2}$ **10.** $\cos^{-1}0$ **11.** $\arccos\left(-\dfrac{\sqrt{2}}{2}\right)$ **12.** $\arcsin(-1)$

13. $\tan^{-1}1$ **14.** $\arctan(-\sqrt{3})$ **15.** $\arccos(-1)$ **16.** $\tan^{-1}\left(-\dfrac{\sqrt{3}}{3}\right)$ **17.** $\arctan 0$

18. $\cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$ **19.** $\sin^{-1}0$ **20.** $\arcsin\left(-\dfrac{\sqrt{3}}{2}\right)$

Find approximate values for the following expressions in both radians and degrees. Round the values for radians to two decimal places and for degrees to one decimal place.

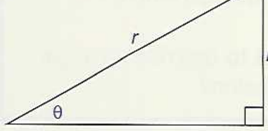
21. $\arctan(-0.2553)$ **22.** $\arccos(-0.9888)$ **23.** $\tan^{-1}11.08$ **24.** $\arccos 0.8253$
25. $\tan^{-1}0.9316$ **26.** $\sin^{-1}0.9323$ **27.** $\sin^{-1}0.6442$ **28.** $\arccos(-0.9902)$
29. $\sin^{-1}(-0.9976)$ **30.** $\arcsin(-0.2955)$ **31.** $\tan^{-1}(-3.4776)$ **32.** $\arcsin(-0.2571)$
33. $\arccos(-0.5299)$ **34.** $\cos^{-1}0.9632$

Simplify the following expressions, obtaining exact values.

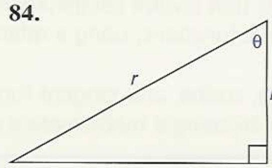
35. $\cos(\arcsin\frac{3}{5})$ **36.** $\tan[\arcsin(-0.8)]$ **37.** $\cos(\arcsin 0.3)$ **38.** $\tan(\arcsin 0.4)$
39. $\cos(\arctan 0.3)$ **40.** $\sin(\arctan 0.4)$ **41.** $\cos(\tan^{-1}(-\frac{3}{5}))$ **42.** $\sin(\cos^{-1}(-\frac{1}{4}))$
43. $\sin(\arccos\frac{5}{8})$ **44.** $\cos(\arctan 2)$ **45.** $\tan[\cos^{-1}(-\frac{2}{3})]$ **46.** $\sin[\arccos(-0.8)]$
47. $\sin(\tan^{-1}\sqrt{5})$ **48.** $\csc\left(\cos^{-1}\dfrac{\sqrt{2}}{6}\right)$ **49.** $\cot\left(\sin^{-1}\dfrac{\sqrt{3}}{5}\right)$ **50.** $\csc\left(\sin^{-1}\dfrac{\sqrt{2}}{6}\right)$
51. $\sec[\sin^{-1}(-\frac{2}{3})]$ **52.** $\tan(\arcsin\frac{5}{8})$ **53.** $\cos(\sin^{-1}z)$, $z > 0$ **54.** $\cos(\sin^{-1}3z)$, $z < 0$
55. $\tan[\sin^{-1}(1 + z)]$, $1 + z < 0$ **56.** $\cos(\arcsin\sqrt{z})$ **57.** $\sec(\arcsin\sqrt{2z})$
58. $\cot(\sin^{-1}\sqrt{z-1})$ **59.** $\sin(\cos^{-1}z)$, $z > 0$ **60.** $\cos(\arccos z)$, $z > 0$ **61.** $\tan(\arccos z)$, $z > 0$
62. $\sin(\tan^{-1}z)$, $z > 0$ **63.** $\cos(\arctan z)$, $z < 0$ **64.** $\tan(\cos^{-1}z)$, $z < 0$ **65.** $\sin(\cos^{-1}3z)$, $z < 0$
66. $\cos(\arctan 2z)$, $z > 0$ **67.** $\sec[\tan^{-1}(1 + z)]$, $z > 0$ **68.** $\sin(\arccos\sqrt{z})$ **69.** $\cos(\arctan\sqrt{2z})$
70. $\tan(\cos^{-1}\sqrt{z-1})$ **71.** $\sin^{-1}\left(\sin\dfrac{\pi}{6}\right)$ **72.** $\arcsin\left(\tan\dfrac{\pi}{4}\right)$ **73.** $\arcsin\left(\cos\dfrac{2\pi}{3}\right)$
74. $\sin^{-1}\left(\sin\dfrac{11\pi}{6}\right)$ **75.** $\sin^{-1}(\cos 0)$ **76.** $\arcsin\left(\sin\dfrac{5\pi}{6}\right)$ **77.** $\tan^{-1}\left(\sin\dfrac{\pi}{2}\right)$
78. $\cos^{-1}\left(\sin\dfrac{7\pi}{6}\right)$ **79.** $\cos^{-1}\left(\cos\dfrac{11\pi}{6}\right)$ **80.** $\tan^{-1}(\cos 0)$ **81.** $\arccos\left(\tan\dfrac{5\pi}{4}\right)$
82. $\cos^{-1}\left(\cos\dfrac{3\pi}{2}\right)$

In the following problems, state the angle θ shown in each diagram in terms of an inverse trigonometric function.

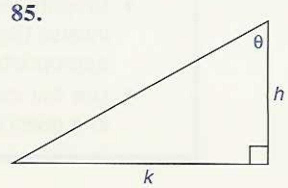
83.



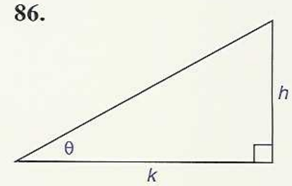
84.



85.



86.



87. A picture hangs on a wall so that the bottom of the picture is 5 feet above the floor. The picture is 2 feet high. Describe the angle determined by the picture at the eye of an observer in terms of an inverse trigonometric function, if the observer's eye is also 5 feet above the floor and the observer is x feet away from the wall.

88. A radar antenna will track the launch of a rocket from a point 12,500 feet from the launch point. Both are at the same ground elevation at launch. If a represents the altitude of the rocket in feet, describe the angle of elevation θ of the rocket at the radar site in terms of an inverse trigonometric function.

89. An aircraft is flying toward an airport at an elevation of 3,500 feet above the airport. Describe the angle of depression θ of the airport at the aircraft in terms of the distance z from the aircraft directly to the airport, using an inverse trigonometric function.

In the following problems describe one value of θ, or x, in exact form, in terms of an inverse trigonometric function.

90. $\sin\theta = 0.75$

91. $\cos\theta = -0.8$

92. $\tan\theta = 3$

93. $\tan\theta = 4.1$

94. $2\sin\theta = 1.6$

95. $3\tan\theta = 5$

96. $\frac{1}{2}\sin\theta = -0.1$

97. $\dfrac{\tan\theta}{5} = 10$

98. $\sin 2\theta = 0.76$

99. $\tan 3\theta = 9$

100. $\cos\dfrac{\theta}{3} = -0.42$

101. $\sin\dfrac{3\theta}{2} = -0.56$

102. $4\cos 3\theta = 3$

103. $2\sin 4\theta = 1.5$

104. $\dfrac{3\cos 2\theta}{8} = \dfrac{7}{40}$

105. $\dfrac{6\sin 5\theta}{5} = \dfrac{10}{13}$

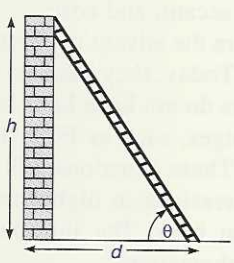
106. $\dfrac{A\cos Bx}{C} = D$

107. $A\tan(Bx + C) = D$

108. $\cos(x - 2) = 0.2$

109. $\sin(2x + 3) = 0.6$

110. Firefighters use the formula $d = \dfrac{h}{4} + 2$ to obtain the safe height of a fire ladder resting against a wall, where h is the height up the wall and d is the distance away from the wall at which the ladder should rest. Let θ represent the angle the ladder makes with the ground. Write the value of θ in terms of the inverse tangent function and the variable h.



111. Some computer languages provide only an arctangent function. In these situations, we must program our own arcsine function. Use appropriate reference triangles to show that the following is an identity.

$$\arcsin(x) = \begin{cases} -\dfrac{\pi}{2} & \text{if } x = -1 \\ \arctan\left(\dfrac{x}{\sqrt{1 - x^2}}\right) & \text{if } |x| < 1 \\ \dfrac{\pi}{2} & \text{if } x = 1 \end{cases}$$

112. Some computer languages provide only an arctangent function. In these situations we must program our own arccosine function. Use appropriate reference triangles to show that the following is an identity.

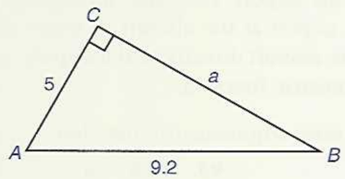
$$\arccos(x) = \begin{cases} \arctan\left(\dfrac{\sqrt{1 - x^2}}{x}\right) & \text{if } 0 < x \le 1 \\ \dfrac{\pi}{2} & \text{if } x = 0 \\ \arctan\left(\dfrac{\sqrt{1 - x^2}}{x}\right) + \pi & \text{if } -1 \le x < 0 \end{cases}$$

Skill and review

1. Perform the indicated multiplication, then simplify the result:

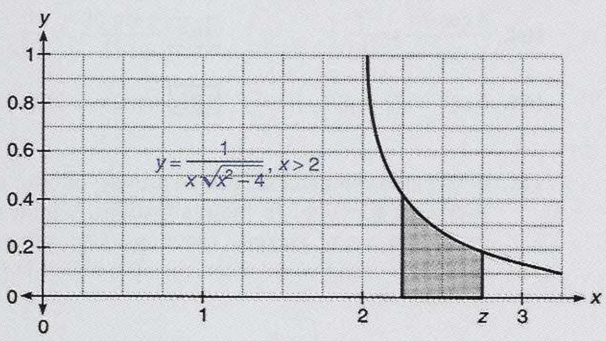
$$\cos x(\cos x + \sin x \tan x - \sec x)$$

2. Solve the triangle in the figure.



3. Find $\sin \frac{7\pi}{6}$.
4. Find $\tan\left(-\frac{11\pi}{3}\right)$.
5. A circle has radius 12 inches. What is the central angle which corresponds to an arc length of 15 inches? State the result in radians (exact) and degrees (nearest 0.1°).
6. Graph the rational function $f(x) = \frac{2x}{x^2 - 4}$.

6–5 The inverse cotangent, secant, and cosecant functions



In higher mathematics it can be shown that the area A between the curve $y = \frac{1}{x\sqrt{x^2 - 4}}$ and the x-axis between $x = 2.25$ and $x = z$, $z > 2$, is $A = \frac{1}{2}\left(\sec^{-1}\frac{z}{2} - \sec^{-1}\frac{2.25}{2}\right)$. Calculate A if $z = 3.25$.

The reciprocal trigonometric functions (cotangent, secant, and cosecant) and their inverses were useful for solving triangles before the advent of electronic calculating devices like calculators and computers. Today, they have no practical use in solving triangles. This is why calculators do not have keys for the reciprocal functions, and why programming languages, such as FORTRAN, BASIC, or Pascal, do not support these functions. These functions still have value in simplifying and transforming certain expressions in higher mathematics, however, and that is why we study them here. The introductory problem illustrates one application from higher mathematics.

The inverse cotangent function

We define the inverse cotangent function by reversing the ordered pairs in the basic cotangent cycle; that is, we restrict the domain to $0 < x < \pi$. Figure 6–28 shows the inverse cotangent function, cotangent^{-1}. The domain is R, and the range is $0 < y < \pi$. As we might suspect *arccot* x also means $\cot^{-1}x$.

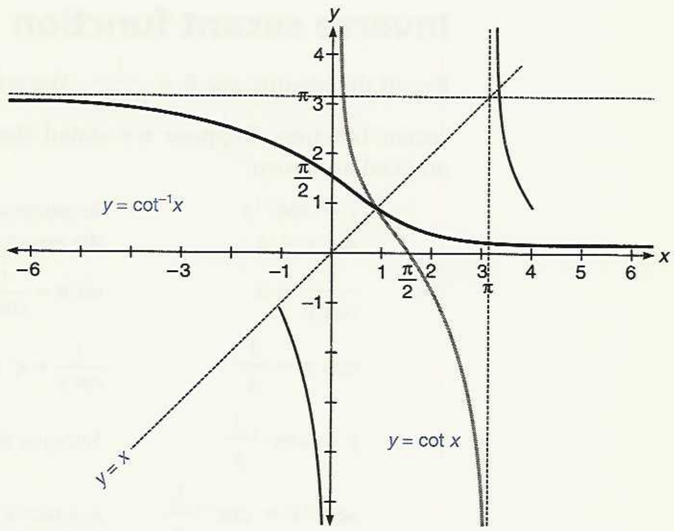


Figure 6–28

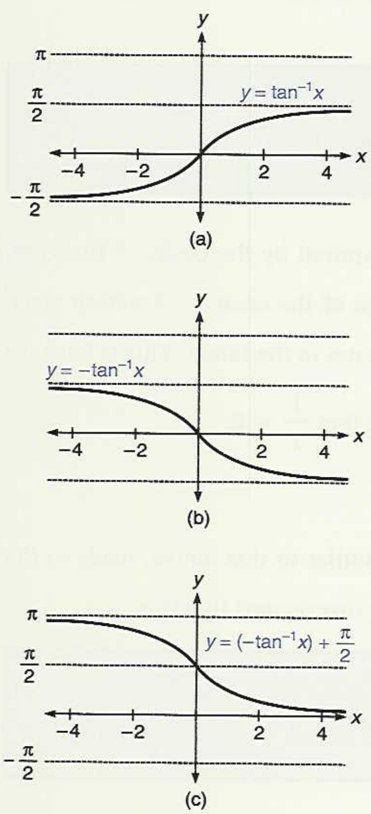


Figure 6–29

> **The inverse cotangent function**
> $y = \cot^{-1}x$ means
> 1. $\cot y = x$
> 2. $0 < y < \pi$

There are several ways to compute values of the inverse cotangent function. One way is to use the identity

$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$$

This identity can be seen by considering the graphs of $y = \cot^{-1}x$ (figure 6–28) and $y = \tan^{-1}x$ (figure 6–27). Figure 6–29 shows the sequence of operations that will transform one graph into the other. Part (a) shows $y = \tan^{-1}x$. Part (b) shows the transformation caused by the scaling factor -1. Part (c) shows the vertical shift caused by adding $\frac{\pi}{2}$. This result is the same as the graph of the inverse cotangent function.

Note that this transformation of graphs does not absolutely guarantee that the identity $\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$ is true. It should be proven algebraically. We will not prove it here.

Inverse secant function

Recall the identity $\sec \theta = \dfrac{1}{\cos \theta}$. We will use this to define the inverse of the secant function. Suppose we stated that $y = \sec^{-1}x$. Then $\sec y = x$. We proceed as shown.

$y = \sec^{-1}x$	An expression we want to define
$\sec y = x$	We expect the expression to have this property
$\dfrac{1}{\cos y} = x$	$\sec \theta = \dfrac{1}{\cos \theta}$
$\cos y = \dfrac{1}{x}$	$\dfrac{1}{\cos y} = x$; $1 = x \cos y$; $\dfrac{1}{x} = \cos y$
$y = \cos^{-1}\dfrac{1}{x}$	Since y is the angle whose cosine is $\dfrac{1}{x}$
$\sec^{-1}x = \cos^{-1}\dfrac{1}{x}$	$y = \sec^{-1}x$

We use this sequence of steps to motivate our definition. As expected, *arcsec* x also means $\sec^{-1}x$.

The inverse secant function

$$\sec^{-1}x = \cos^{-1}\frac{1}{x} \quad \text{if } |x| \geq 1$$

We require $|x| \geq 1$ so that $\left|\dfrac{1}{x}\right| \leq 1$, as required by the cosine^{-1} function. The range of the secant function is the range of the cosine^{-1} function since the latter defines the former, *except* that $\dfrac{\pi}{2}$ is not in the range. This is because $\dfrac{\pi}{2} = \cos^{-1}0$, and there is no value of x such that $\dfrac{1}{x} = 0$.

Inverse cosecant function

The identity $\csc \theta = \dfrac{1}{\sin \theta}$, and reasoning similar to that above, leads to the following definition for the inverse cosecant (*arcsecant*) function.

The inverse cosecant function

$$\csc^{-1}x = \sin^{-1}\frac{1}{x} \quad \text{if } |x| \geq 1$$

For reasons similar to those stated for the inverse secant function, the domain of the cosecant^{-1} function is $|x| \geq 1$, and the range is the same as that of the sine^{-1} function, except for 0, which is $\sin^{-1}0$, and $\frac{1}{x}$ can not take on the value 0.

Summary of properties

Table 6–5 summarizes the domains and ranges of the functions introduced above, as well as the identity mentioned above. Note in table 6–5 that quadrant I always corresponds to a nonnegative domain element (a nonnegative value of x), and quadrant II or IV corresponds to negative domain elements.

Example 6–5 A illustrates computations.

Function	Domain	Range	Quadrants
$y = \cot^{-1}x$	R	$0 < y < \pi$	I, II
$y = \sec^{-1}x$	$\|x\| \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$	I, II
$y = \csc^{-1}x$	$\|x\| \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$	I, IV

$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$$

Table 6–5

■ ***Example 6–5 A***

Find the given values in both radians and degrees. Round radians to two decimal places and degrees to one decimal place where necessary.

1. $\cot^{-1}(-4)$

$= \frac{\pi}{2} - \tan^{-1}(-4)$ — $\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

≈ 2.90 (radians) — Calculator in radian mode

$= 90° - \tan^{-1}(-4)$

$\approx 166.0°$ — Calculator in degree mode

Thus, $\cot^{-1}(-4) \approx 166.0°$ or 2.90 (radians).

2. $\sec^{-1}2$

$= \cos^{-1}\frac{1}{2}$ — Definition of $\sec^{-1}\theta$

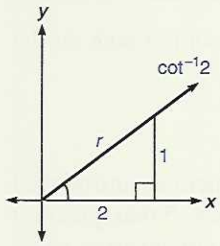
$= 60°$ or $\frac{\pi}{3}$

Thus, $\sec^{-1}2 = 60°$ or $\frac{\pi}{3}$. ■

As we saw in section 6–4, some expressions that involve both the trigonometric and inverse trigonometric functions can be simplified by using a reference triangle. Example 6–5 B illustrates for the functions covered in this section.

■ *Example 6–5 B*

Simplify the expression.

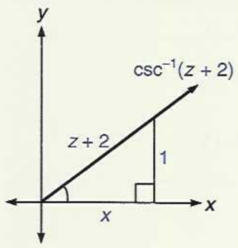


1. $\sec(\cot^{-1}2)$

$\cot^{-1}2$ is an angle in quadrant I. Since its cotangent is 2, its tangent is $\frac{1}{2}$. A reference triangle for a quadrant I angle with tangent $\frac{1}{2}$ is shown in the figure.

$r = \sqrt{5}$ (Pythagorean theorem), so the cosine of the angle is $\dfrac{2}{\sqrt{5}}$, and therefore the secant is $\dfrac{\sqrt{5}}{2}$. Thus, $\sec(\cot^{-1}2) = \dfrac{\sqrt{5}}{2}$.

Note The identity $\cot^{-1}x = \dfrac{\pi}{2} - \tan^{-1}x$ would not be helpful in simplifying this expression. It is useful for computing values of $\cot^{-1}x$.



2. $\cos(\csc^{-1}(z + 2)),\ z > 0$

$\csc^{-1}(z + 2) = \sin^{-1}\dfrac{1}{z + 2}$ by definition. Since $\dfrac{1}{z + 2}$ is positive, $\sin^{-1}\dfrac{1}{z + 2}$ is an angle in quadrant I (see the figure). We find x by the Pythagorean theorem.

$$(z + 2)^2 = 1^2 + x^2$$
$$z^2 + 4z + 4 - 1 = x^2$$
$$\sqrt{z^2 + 4z + 3} = x$$

The cosine of the angle is $\dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{x}{z + 2} = \dfrac{\sqrt{z^2 + 4z + 3}}{z + 2}$. ■

Mastery points

Can you

- State the domain and range of the inverse cotangent, cosecant, and secant functions?
- Find exact values, in both radians and degrees, for expressions of the form $\cot^{-1}x$, $\csc^{-1}x$, and $\sec^{-1}x$, for appropriate values of x, using the definitions of these functions?
- Find approximate values, in both radians and degrees, for expressions of the form $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$, using a calculator and the definitions of these functions?
- Simplify expressions that involve combinations of the trigonometric and inverse trigonometric functions, using a reference triangle where appropriate?

Exercise 6–5

Compute the exact value of the following expressions.

1. $\csc^{-1}2$
2. $\text{arcsec}\left(-\frac{2\sqrt{3}}{3}\right)$
3. arccot 1
4. arccsc(−1)
5. $\sec^{-1}(-2)$
6. $\cot^{-1}(-\sqrt{3})$
7. $\text{arccsc}\left(\frac{2\sqrt{3}}{3}\right)$
8. $\csc^{-1}\sqrt{2}$
9. arccot 0
10. $\text{arcsec}(-\sqrt{2})$

Compute approximate values of the following expressions in both radians (to hundredths) and degrees (to tenths).

11. $\csc^{-1}3.3534$
12. $\cot^{-1}0.5080$
13. arcsec(−2.9986)
14. arccsc(−2.5087)
15. arccot 5.1997
16. $\sec^{-1}(-2.0126)$
17. $\sec^{-1}(-11.1261)$
18. $\csc^{-1}(-3.8898)$
19. arccsc 3.1790
20. arccot(−8.3534)

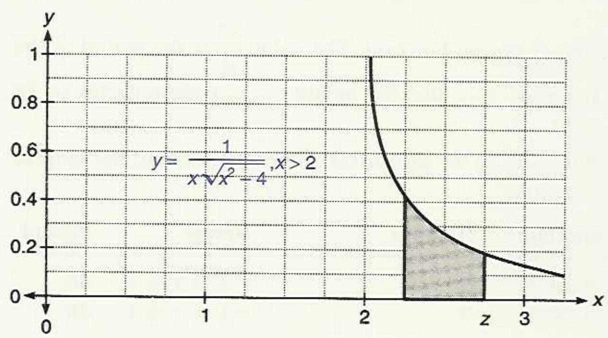
Simplify the following expressions.

21. $\sin(\csc^{-1}3)$
22. $\cos(\cot^{-1}2)$
23. cot(arcsec 4)
24. $\sin(\sec^{-1}1.5)$
25. csc(arccot 5)
26. $\sec(\text{arccsc}\ \frac{7}{4})$
27. $\cos[\text{arccsc}(-\frac{5}{4})]$
28. $\tan[\text{arccsc}(-\frac{5}{3})]$
29. $\tan[\sec^{-1}(-\frac{6}{5})]$
30. $\csc[\sec^{-1}(-\frac{8}{7})]$
31. $\sin(\csc^{-1}z)$, $z > 0$
32. $\cot(\sec^{-1}z)$, $z > 0$
33. $\sec(\cot^{-1}z)$, $z < 0$
34. $\tan(\text{arcsec}\ z)$, $z < 0$
35. $\cos(\text{arcsec}\ 2z)$, $z > 0$
36. $\sin(\csc^{-1}3z)$, $z < 0$
37. $\tan[\sec^{-1}(z + 1)]$, $z + 1 > 0$
38. $\csc[\sec^{-1}(1 - z)]$, $1 - z > 0$
39. $\csc\left[\text{arccsc}\ \frac{3}{z}\right]$, $z > 0$
40. $\sec\left[\cot^{-1}\left(\frac{2}{z+1}\right)\right]$, $z + 1 > 0$

41. In higher mathematics it can be shown that the area A between the curve $y = \frac{1}{x\sqrt{x^2 - 4}}$ and the x-axis between $x = 2.25$ and $x = z$, $z > 2$, is (see the figure)

$$A = \frac{1}{2}\left(\sec^{-1}\frac{z}{2} - \sec^{-1}\frac{2.25}{2}\right)$$

Calculate A, to the nearest thousandth, if $z = 3.25$.



Skill and review

1. Combine: $\frac{2x}{x-3} - \frac{x}{x+5}$.
2. If $\sin x = -\frac{1}{2}$ and x terminates in quadrant III, find the measure of x in radians (exact).
3. If $\cos\theta = 1 - u$, and θ terminates in quadrant II, find $\sin\theta$ in terms of u.
4. Find the least nonnegative solution to the equation $3\sin x = -2$, to the nearest 0.1°.
5. Simplify $\cos(\tan^{-1}5)$.
6. Simplify $\csc(\cos^{-1}(1 - m))$, $1 - m > 0$.
7. Find $\csc\left(\frac{19\pi}{6}\right)$.
8. If the point (−5,8) is on the terminal side of an angle θ in standard position, (a) find $\sin\theta$ and (b) find the least nonnegative measure of θ to the nearest 0.1°.

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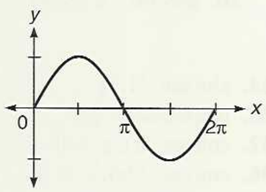
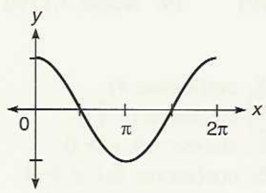
Chapter 6 summary

- Let θ be an angle in standard position with degree measure $\theta°$ and radian measure s. Then, $\dfrac{s}{\pi} = \dfrac{\theta°}{180°}$.
- If (x,y) is the point on the unit circle that intersects the terminal side of θ, then $\sin\theta = y$, $\cos\theta = x$.
- If s is the radian measure of a central angle on a circle of radius r, and L is the corresponding arc length, then $L = rs$.



Basic sine cycle



Basic cosine cycle

- The expression $-\dfrac{C}{B}$ is the phase shift, of the particular sine or cosine function. The value $\dfrac{2\pi}{B}$ is the period of the function. B is also the number of complete cycles in 2π units.
- Summary of the properties of the sine, cosine, and tangent functions.

Function	Domain	Range	Period
$y = \sin x$	R	$-1 \le y \le 1$	2π
$y = \cos x$	R	$-1 \le y \le 1$	2π
$y = \tan x$	$x \ne \dfrac{\pi}{2} + k\pi,\ k \in J$	R	π

$\sin(-x) = -\sin x$ (odd, origin symmetry)
$\cos(-x) = \cos x$ (even, y-axis symmetry)
$\tan(-x) = -\tan x$ (odd, origin symmetry)

- Summary of the properties of the cosecant, secant, and cotangent functions.

Function	Domain	Range	Period
$y = \csc x$	$x \ne k\pi,\ k \in J$	$\lvert y\rvert \ge 1$	2π
$y = \sec x$	$x \ne \dfrac{\pi}{2} + k\pi,\ k \in J$	$\lvert y\rvert \ge 1$	2π
$y = \cot x$	$x \ne k\pi,\ k \in J$	R	π

$\csc(-x) = -\csc x$ (odd, origin symmetry)
$\sec(-x) = \sec x$ (even, y-axis symmetry)
$\cot(-x) = -\cot x$ (odd, origin symmetry)

- **Inverse sine function** $y = \sin^{-1}x$ means
  1. $\sin y = x$
  2. $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$
  3. $-1 \le x \le 1$
- **Inverse cosine function** $y = \cos^{-1}x$ means
  1. $\cos y = x$
  2. $0 \le y \le \pi$
  3. $-1 \le x \le 1$
- **Inverse tangent function** $y = \tan^{-1}x$ means
  1. $\tan y = x$
  2. $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$
- **Inverse cotangent function** $y = \cot^{-1}x$ means
  1. $\cot y = x$
  2. $0 < y < \pi$
- **Inverse secant function** $\sec^{-1}x = \cos^{-1}\dfrac{1}{x}$ if $\lvert x\rvert \ge 1$.
- **Inverse cosecant function** $\csc^{-1}x = \sin^{-1}\dfrac{1}{x}$ if $\lvert x\rvert \ge 1$.
- Summary of the properties of the inverse sine, cosine, and tangent functions.

Function	Domain	Range	Quadrants
$y = \sin^{-1}x$	$-1 \le x \le 1$	$-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$	I, IV
$y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$	I, II
$y = \tan^{-1}x$	R	$-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$	I, IV

$\sin^{-1}(-x) = -\sin^{-1}x$ (odd)
$\cos^{-1}(-x) = \pi - \cos^{-1}x$
$\tan^{-1}(-x) = -\tan^{-1}x$ (odd)

- Summary of the properties of the inverse cotangent, secant, and cosecant functions.

Function	Domain	Range	Quadrants
$y = \cot^{-1}x$	R	$0 < y < \pi$	I, II
$y = \sec^{-1}x$	$\lvert x\rvert \ge 1$	$0 \le y \le \pi,\ y \ne \dfrac{\pi}{2}$	I, II
$y = \csc^{-1}x$	$\lvert x\rvert \ge 1$	$-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2},\ y \ne 0$	I, IV

$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$$

Chapter 6 review

[6–1] Convert the following degree measures to radian measures. Leave answers in both exact form and approximated to two decimal places.

1. 315° **2.** −240° **3.** −148° **4.** 320°

Convert the following radian measures into degree measures. Leave answers both in exact form and approximated to two decimal places.

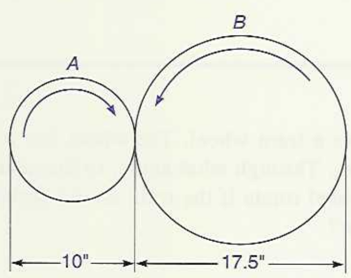
5. $\frac{7\pi}{9}$ **6.** $\frac{11\pi}{8}$ **7.** −4 **8.** $\frac{3}{8}$

9. Find the length of the arc determined by a central angle of 3.8 radians on a circle of diameter 16 inches.

10. Find the length of the arc determined by a central angle of 230° on a circle of radius 150 millimeters.

11. Find the measure, in both radians and degrees, of the central angle determined by an arc length of 32 inches on a circle of diameter 20 inches.

12. The diagram shows two wheels. Wheel A drives wheel B. Determine the radian measure of the angle through which wheel B will move when wheel A moves through an angle of 5 radians.



Find the following function values where the angle is given in radian measure. Round your answer to four decimal places.

13. sin 4 **14.** cos 2.52 **15.** sec $\frac{3}{8}$ **16.** cot 1

Find the exact function values for the following angles.

17. $\cos\frac{5\pi}{6}$ **18.** $\cot\frac{5\pi}{3}$

[6–2]

19. Sketch the graph of the sine function; state the domain, range, and period of the sine function.

20. Using the graph of $y = \cos x$ as a guide describe all values of x for which $\cos x$ is 1.

Use the appropriate property, even or odd, to calculate the exact function value.

21. $\sin\left(-\frac{\pi}{3}\right)$ **22.** $\tan\left(-\frac{4\pi}{3}\right)$

Test the function for the even/odd property. State which type of symmetry the graph would have based on being even, odd, or neither even nor odd.

23. $f(x) = x \sin x$ **24.** $f(x) = \tan x \cdot \cos x$

Graph three cycles of the following functions.

25. $y = -3 \sin x$ **26.** $y = 2 \cos x - 1$

27. $y = -\tan x$ **28.** $y = \sin 4x$

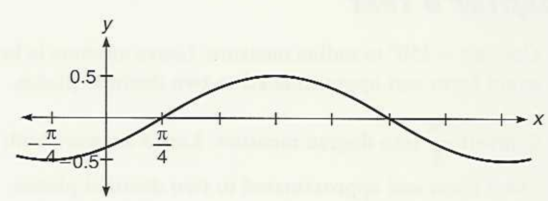
29. $y = \cos 3x$ **30.** $y = \tan\frac{x}{3}$

31. $y = \frac{1}{2}\sin\left(x - \frac{\pi}{4}\right)$ **32.** $y = 2\cos\left(x + \frac{\pi}{6}\right)$

33. $y = \cos 2\pi x$ **34.** $y = 3\sin(\pi - 2x)$

35. $y = 2\cos(\pi - 3x)$

36. Assume that the graph shown is of the form $y = A\sin(Bx + C) + D$. Find values of A, B, C, and D that would produce the graph, and state the equation produced by these values.



[6–3]

37. Sketch the graph of the cosecant function.

38. Show that the function $f(x) = \sec x \cdot \sin^2 x + x^4$ is an even function.

Graph three cycles of the following functions.

39. $y = \tan 3x$ **40.** $y = \sec\left(x + \frac{\pi}{3}\right)$

41. $y = \tan(-x)$

[6–4]

42. State the domain and range of the inverse cosine function.

Find exact values for each of the following expressions, in both radians and degrees.

43. $\sin^{-1}\frac{1}{2}$ **44.** $\cos^{-1}\dfrac{\sqrt{2}}{2}$ **45.** $\arctan\sqrt{3}$

46. $\arcsin\left(-\dfrac{\sqrt{3}}{2}\right)$

Find approximate values for the following expressions in both radians and degrees. Round the values for radians to two decimal places and for degrees to one decimal place.

47. $\arctan(-3.55)$ **48.** $\arccos(-0.4290)$

49. $\tan^{-1}2.5$

Simplify the following expressions. Obtain exact values.

50. $\cos(\arcsin\frac{3}{8})$ **51.** $\tan[\arcsin(-\frac{2}{3})]$

52. $\cos(\arcsin 0.5)$ **53.** $\csc\left(\cos^{-1}\dfrac{\sqrt{2}}{4}\right)$

54. $\sin[\cos^{-1}(-\frac{1}{3})]$ **55.** $\sin[(\arctan(-5))]$

56. $\cos(\sin^{-1}2z)$, $z > 0$ **57.** $\tan[\sin^{-1}(1 - z)]$, $1 - z < 0$

58. $\sin(\arccos\sqrt{z + 1})$ **59.** $\sec(\tan^{-1}\sqrt{z - 1})$

60. $\sin^{-1}\left(\sin\dfrac{7\pi}{6}\right)$ **61.** $\sin^{-1}\left(\cos\dfrac{\pi}{2}\right)$

62. $\arccos\left(\cos\dfrac{5\pi}{3}\right)$

63. An aircraft is flying toward an airport at an elevation of 6,000 feet above the airport. Describe the angle of depression of the airport at the aircraft in terms of the slant distance z from the aircraft directly to the airport, using an inverse trigonometric function.

In the following problems, describe one value of θ, or x, in exact form, in terms of an inverse trigonometric function.

64. $\cos\dfrac{\theta}{3} = \dfrac{1}{3}$ **65.** $\frac{1}{3}\sin 2x = \frac{1}{12}$

66. $a\tan k\theta = b$ (a, b, and k are constants).

[6–5] Compute the exact value of the following expressions.

67. $\csc^{-1}2$ **68.** $\sec^{-1}\sqrt{2}$

Find approximate values for the following expressions in both radians and degrees. Round the values for radians to two decimal places and for degrees to one decimal place.

69. $\cot^{-1}3$ **70.** $\text{arcsec}(-2.66)$

Simplify the following expressions.

71. $\cot(\sec^{-1}3)$ **72.** $\sec\left(\cot^{-1}\dfrac{1}{z}\right)$, $z > 0$

Chapter 6 test

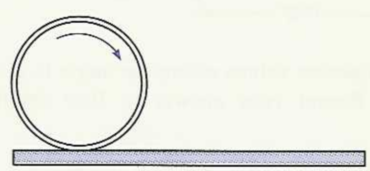
1. Convert $-250°$ to radian measure. Leave answers in both exact form and approximated to two decimal places.

2. Convert $\dfrac{7\pi}{5}$ into degree measure. Leave answers both in exact form and approximated to two decimal places.

3. Find the length of the arc, to the nearest tenth, determined by a central angle of 2.5 radians on a circle of diameter 20 inches.

4. Find the measure, in both radians and degrees, to the nearest tenth, of the central angle determined by an arc length of 19 millimeters on a circle of diameter 28 millimeters.

5. The diagram shows a train wheel. The wheel has a diameter of 38 inches. Through what angle, to the nearest degree, will the wheel rotate if the train moves forward a distance of 5 feet?



6. Find the value of sec 3.12 to four decimal places.

7. Find the exact function value of $\cos\dfrac{11\pi}{6}$.

8. Sketch the graph of the cosine function; state the domain, range, and period of the cosine function.

9. Using the graph of $y = \sin x$ as a guide describe all values of x for which $\sin x$ is -1.

10. Use the appropriate property, even or odd, to calculate the exact function value of $\tan\left(-\frac{5\pi}{3}\right)$.

11. Test the function $f(x) = x + \sin x$ for the even/odd property. State which type of symmetry the graph would have based on being even, odd, or neither even nor odd.

Graph three cycles of the following functions.

12. $y = -\frac{1}{2}\sin x$

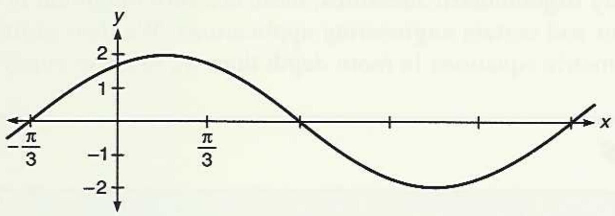
13. $y = 3\cos x + 1$

14. $y = \sin 3x$

15. $y = \cos\left(x + \frac{\pi}{6}\right)$

16. $y = 3\sin(\pi - 2x)$

17. Assume that the graph is of the form $y = A\sin(Bx + C) + D$. Find values of A, B, C, and D that would produce the graph, and write the corresponding equation.



18. Sketch the graph of the secant function.

19. Show that the function $f(x) = \sec x \cdot \sin x + x^3$ is an odd function.

Graph three cycles of the following functions.

20. $y = \tan(-2x)$

21. $y = \csc\left(x - \frac{\pi}{6}\right)$

22. State the domain and range of the inverse cosine function.

23. Find the exact value for $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ both in radians and degrees.

24. Find an approximate value for $\cos^{-1}(-0.80)$ in both radians and degrees. Round the values for radians to two decimal places and for degrees to one decimal place.

Simplify the following expressions. Obtain exact values.

25. $\tan(\arcsin \frac{3}{4})$

26. $\tan[\arccos(-\frac{1}{3})]$

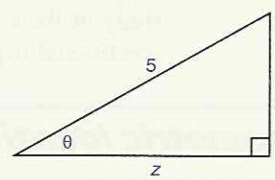
27. $\csc\left(\sin^{-1}\frac{\sqrt{3}}{4}\right)$

28. $\cos(\tan^{-1}2z)$, $z > 0$

29. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

30. $\csc^{-1}\frac{2}{\sqrt{3}}$

31. Write angle θ in the triangle shown in terms of z and 5, using an inverse trigonometric function.



32. Describe one value of x in exact form, in terms of an inverse trigonometric function: $2\cos 3x = \frac{1}{3}$.

33. Find approximate values for the following expression in both radians and degrees. Round the values for radians to two decimal places and for degrees to one decimal place: $\sec^{-1}2.65$.